A class of time discrete schemes for a phase–field system of Penrose–Fife type

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Abstract

In this paper, a phase field system of Penrose–Fife type with non–conserved order parameter is considered. A class of time–discrete schemes for an initial–boundary value problem for this phase–field system is presented. In three space dimensions, convergence is proved and an error estimate is derived. For one scheme, this error estimate is linear with respect to the time–step size.

1 Introduction

In [PF90], Penrose and Fife derived a phase–field system modeling the dynamics of diffusive phase transitions. In the case of a non–conserved order parameter, their approach leads to the following system:

$$c_0\theta_t + \lambda'(\chi)\chi_t + \kappa\Delta\left(\frac{1}{\theta}\right) = g, \tag{1.1}$$

$$\eta\chi_t - \varepsilon\Delta\chi + \beta(\chi) - \sigma'(\chi) \ni -\frac{\lambda'(\chi)}{\theta}. \tag{1.2}$$

This system determines the evolution of the order parameter χ and the absolute temperature θ. Here, c_0 and κ denote the physical data specific heat and thermal conductivity, which are supposed to be positive constants. The datum g represents heat sources or sinks, and η stands for a positive space–dependent relaxation coefficient. Choosing this coefficient in a particular way, an anisotropic growth can be simulated.
ε is a positive relaxation coefficient and β denotes the subdifferential of the convex but non–smooth part of a potential on $\mathbb{R}$, while $-\sigma$ corresponds to the non–convex but differentiable part of the potential. The latent heat of the phase transition is represented by $\lambda'(\chi)$.
In the context of solid–liquid phase transitions, one typically has a quadratic or linear function λ and

$$\sigma(s) = \frac{\lambda(s)}{\theta_C} + \rho s^2, \quad \forall\, s \in \mathbb{R}, \tag{1.3}$$

where θ_C denotes some critical temperature and ρ some positive constant. For $\beta(s) = 2\rho s^3$, we see that $\beta(s) - \sigma'(s) + \theta_C^{-1}\lambda'(s)$ is the derivative of the *double well potential* $\frac{\rho}{2}(s-1)^2(s+1)^2$. If β is the subdifferential of the indicator function $I_{[-1,1]}$ of the interval $[-1,1]$, we see that $\beta(s) - \sigma'(s) + \theta_C^{-1}\lambda'(s)$ corresponds to the "derivative" of the *double obstacle potential* $I_{[-1,1]}(s) + \rho(1-s^2)$, which has been introduced for the standard phase–field system by Blowey and Elliott (see [BE94]).
In the mean–field theory of the Ising ferromagnet as in [PF90, Sec. 4], one has quadratic functions σ and λ, $D(\beta) = (0,1)$, and

$$\beta(s) = \rho^*\frac{\partial}{\partial s}\left(s\ln s + (1-s)\ln(1-s) - \ln\left(\frac{1}{2}\right)\right) = \rho^*\ln\left(\frac{s}{1-s}\right), \quad \forall\, s \in D(\beta),$$

where ρ^* is some positive constant.

The results in this work cover all these situations. Its main novelty is a time–discrete scheme for an initial–boundary value problem for the phase–field system (1.1)–(1.2) such that in three space dimensions an error estimate linear with respect to the time–step size h can be derived. Moreover, a general class of time–discrete schemes is investigated, including some which are explicit in the approximation of $\sigma'(\chi)$ or $\lambda'(\chi)$. For these schemes an error estimate is derived, which is linear with respect to h in two space dimensions and still nearly linear in three space dimensions.
In [Hor93], Horn considers a time–discrete scheme in one space dimension for the Penrose–Fife system for quadratic λ and σ. He derives an error estimate of order $\sqrt{h}$.
In previous works [Kle97a, Kle97b] of the author a time discrete scheme for a simplified Penrose–Fife system with λ linear and σ linear or quadratic has been considered and an error estimate of order $\sqrt{h}$ has been shown.

Using the time–discrete scheme, the existence of a unique solution to the Penrose–Fife system is proved. This result is a minor novelty of this paper, because of the weakened regularity assumption used for λ and σ. These functions are supposed to be C^1–functions on $\mathbb{R}$ with λ' and σ' locally Lipschitz continuous such that the Lipschitz constants fulfill some growth conditions.
Until now, in papers concerning existence, uniqueness, and regularity of similar Penrose–Fife systems these functions are supposed to be at least C^2–functions with λ'' bounded (see, e.g. [HLS96, HSZ96, Lau93, Lau95, SZ93] or C^1–functions with λ' global Lipschitz (see [KN94]) resp. λ convex (see [DK96]).
The same holds for papers like [CL98, CS98, CLS, Lau98], where more general heat flux laws are considered.

The layout of this paper is as follows: In Section 2, a precise formulation of the considered phase–field system is given, the class of time–discrete schemes is introduced, and the existence and approximation results are presented. The remaining sections are devoted to the proof of these results.
In Section 3, estimates concerning the approximation of the data are derived and the existence of a solution to the scheme is shown under the additional assumption that the domain $D(\beta)$ is bounded. Uniform estimates for a solution to the scheme are derived in Section 4.
Based on these results, the existence of a unique solution to the scheme is proved in Section 5. This is done by considering the time–discrete scheme with β replaced by $\beta + \partial I_{[-C,C]}$, where $I_{[-C,C]}$ denotes the indicator function of the interval $[-C, C]$ for some sufficiently big $C > 0$.
In Section 6, the error estimates are derived, and the existence of a unique solution to the Penrose–Fife system is proved.

2 The Penrose–Fife system and the time–discrete schemes

In this section, a precise formulation of the considered phase–field system of Penrose–Fife type is given. Moreover, existence results and approximation results for a class of time–discrete schemes are presented.

2.1 The phase–field system

In the sequel, $\Omega \subset \mathbb{R}^N$ with $N \in \{2,3\}$ denotes a bounded, open domain with smooth boundary Γ and $T > 0$ stands for a final time. Let $\Omega_T := \Omega \times (0,T)$ and $\Gamma_T := \Gamma \times (0,T)$. We consider the following Penrose–Fife system:

$\boxed{\textbf{(PF):}}$ **Find a quadruple (θ, u, χ, ξ) fulfilling**

$$\theta \in H^1(0,T;L^2(\Omega)), \quad u \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)), \tag{2.1a}$$

$$\chi \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^2(\Omega)), \tag{2.1b}$$

$$\xi \in L^\infty(0,T;L^2(\Omega)), \tag{2.1c}$$

$$\theta > 0, \quad u = \frac{1}{\theta}, \quad \chi \in D(\beta), \quad \xi \in \beta(\chi) \quad \text{a.e. in} \quad \Omega_T, \tag{2.1d}$$

$$c_0\theta_t + \lambda'(\chi)\chi_t + \kappa\Delta u = g \quad \text{a.e. in} \quad \Omega_T, \tag{2.1e}$$

$$\eta\chi_t - \varepsilon\Delta\chi + \xi - \sigma'(\chi) = -\lambda'(\chi)u \quad \text{a.e. in} \quad \Omega_T, \tag{2.1f}$$

$$\kappa\frac{\partial u}{\partial n} + \gamma u = \zeta, \quad \frac{\partial \chi}{\partial n} = 0 \quad \text{a.e. in} \quad \Gamma_T, \tag{2.1g}$$

$$\theta(\cdot,0) = \theta^0, \quad \chi(\cdot,0) = \chi^0 \quad \text{a.e. in} \quad \Omega. \tag{2.1h}$$

For dealing with this system, the following assumptions will be used:

(A1): Let β be a maximal monotone graph on $\mathbb{R}$ and $\phi : \mathbb{R} \to [0,\infty]$ a convex, lower semicontinuous function $\phi : \mathbb{R} \to [0,\infty]$ satisfying

$$\beta = \partial\phi, \quad 0 \in D(\beta), \quad 0 \in \beta(0), \quad \text{int } D(\beta) \neq \emptyset.$$

(A2): There are positive constants C_1^*, p, q such that

$$\lambda \in C^1(\mathbb{R}), \quad \sigma \in C^1(\mathbb{R}), \quad p < 1, \quad q < 4,$$
$$-\lambda(s) \le C_1^*(\phi(s)+1), \quad (\sigma'(s))^2 \le C_1^*(\phi(s)+1), \quad \forall\, s \in D(\beta),$$
$$|\lambda'(s) - \lambda'(r)| \le |s-r|\, C_1^*\left(|s|^p + |r|^p + 1\right), \qquad \forall\, s,r \in D(\beta),$$
$$|\sigma'(s) - \sigma'(r)| \le |s-r|\, C_1^*\left(|s|^q + |r|^q + 1\right), \quad \forall\, s,r \in D(\beta).$$

(A3): We have positive constants c_η, c_γ, and c_ζ such that

$$g \in H^1(0,T;L^\infty(\Omega)), \quad \eta \in L^\infty(\Omega), \quad \eta \ge c_\eta \quad \text{a.e. in} \quad \Omega,$$
$$\gamma \in L^\infty(0,T;C^1(\Gamma)), \quad \gamma_t \in L^\infty(\Gamma_T), \quad \gamma \ge c_\gamma \quad \text{a.e. in} \quad \Gamma_T,$$
$$\zeta \in H^1(0,T;L^2(\Gamma)) \cap L^\infty(\Gamma_T) \cap L^\infty(0,T;H^{\frac{1}{2}}(\Gamma)), \quad \zeta \ge c_\zeta \quad \text{a.e. in} \quad \Gamma_T.$$

(A4): We consider initial data $\theta^0, \chi^0, u^0, \xi^0$ such that

$$\theta^0, u^0 \in H^1(\Omega) \cap L^\infty(\Omega), \quad \chi^0 \in H^2(\Omega), \quad \xi^0 \in L^2(\Omega), \quad \phi(\chi^0) \in L^1(\Omega),$$
$$\theta^0 > 0, \quad u^0 = \frac{1}{\theta^0}, \quad \chi^0 \in D(\beta), \quad \xi^0 \in \beta(\chi^0) \quad \text{a.e. in} \quad \Omega, \quad \frac{\partial\chi^0}{\partial n} = 0 \quad \text{a.e. in} \quad \Gamma.$$

2.2 The class of time discrete schemes

To allow for variable time–steps, we consider decompositions of $(0, T)$ that do not need to be uniform, but satisfy the following assumption.

(A5): The decomposition $Z = \{t_0, t_1, \dots, t_K\}$ with $0 = t_0 < t_1 < \cdots < t_K = T$ fulfills

$$0.01(t_m - t_{m-1}) \le t_{m+1} - t_m \le 2(t_m - t_{m-1}), \quad \forall\, 1 \le m < K. \tag{2.2}$$

Remark 2.1. In the estimate (2.2), the first constant could be replaced by any positive constant smaller than one and the second by any constant bigger than one.

We define the *width* $|Z|$ of the decomposition by $|Z| := \max\limits_{1\le m\le K}(t_m - t_{m-1})$, and, for $1 \le m \le K$,

$$h_m := t_m - t_{m-1}, \qquad\qquad g_m(x) := \frac{1}{h_m}\int\limits_{t_{m-1}}^{t_m} g(x,t)\,\mathrm{d}t\,, \qquad \forall x \in \Omega, \tag{2.3a}$$

$$\gamma_m(\sigma) := \frac{1}{h_m}\int\limits_{t_{m-1}}^{t_m} \gamma(\sigma,t)\,\mathrm{d}t\,, \qquad \zeta_m(\sigma) := \frac{1}{h_m}\int\limits_{t_{m-1}}^{t_m} \zeta(\sigma,t)\,\mathrm{d}t\,, \quad \forall \sigma \in \Gamma. \tag{2.3b}$$

Now, the following time–discrete scheme $(\mathbf{D_Z})$ for the Penrose–Fife system is considered
$\boxed{(\mathbf{D_Z}):}$ **For $1 \le m \le K$, find**

$$\theta_m \in L^2(\Omega), \qquad u_m, \chi_m \in H^2(\Omega), \qquad \xi_m \in L^2(\Omega) \tag{2.4a}$$

such that

$$0 < u_m, \quad \theta_m = \frac{1}{u_m}, \quad \chi_m \in D(\beta), \quad \xi_m \in \beta(\chi_m) \quad \text{a.e. in} \quad \Omega, \tag{2.4b}$$

$$c_0\frac{\theta_m - \theta_{m-1}}{h_m} + \lambda_d'(\chi_m, \chi_{m-1})\frac{\chi_m - \chi_{m-1}}{h_m} + \kappa\Delta u_m = g_m \quad \text{a.e. in} \quad \Omega, \tag{2.4c}$$

$$\eta\frac{\chi_m - \chi_{m-1}}{h_m} - \varepsilon\Delta\chi_m + \xi_m - \sigma_d'(\chi_m, \chi_{m-1}) = -\lambda_d'(\chi_m, \chi_{m-1})\,u_m \quad \text{a.e. in} \quad \Omega, \tag{2.4d}$$

$$-\kappa\frac{\partial u_m}{\partial n} = \gamma_m u_m - \zeta_m, \quad \frac{\partial \chi_m}{\partial n} = 0 \quad \text{a.e. in} \quad \Gamma, \tag{2.4e}$$

with

$$\theta_0 := \theta^0, \quad u_0 := u^0, \quad \chi_0 := \chi^0, \quad \xi_0 := \xi^0. \tag{2.4f}$$

Here, approximations λ_d' and σ_d' for λ' and σ' are used such that the following assumption is satisfied:

(A6): Let $\lambda_d', \sigma_d' : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be continuous functions, and let C_2^*, p, q be positive constants with $p<1$, $q<4$ such that, for all $r,s,r',s'\in D(\beta)$,

$$
\begin{aligned}
\lambda_d'(s,s) = \lambda'(s), \quad \sigma_d'(s,s) = \sigma'(s), \quad (\sigma_d'(r,s))^2 \le C_2^*(\phi(r)+\phi(s)+1),\\
|\lambda_d'(r,r') - \lambda_d'(s,s')| \le C_2^*\left(|r-s|+|r'-s'|\right)\left(|r|^p+|r'|^p+|s|^p+|s'|^p+1\right),\\
|\sigma_d'(r,r') - \sigma_d'(s,s')| \le C_2^*\left(|r-s|+|r'-s'|\right)\left(|r|^q+|r'|^q+|s|^q+|s'|^q+1\right),\\
-\lambda_d'(r,s)(r-s) \le -\lambda(r)+\lambda(s)+C_2^*(r-s)^2.
\end{aligned}
\tag{2.5}
$$

Remark 2.2. The time–discrete scheme $(\mathbf{D_Z})$ is an Euler scheme in time for the Penrose–Fife system **(PF)**, which is fully implicit, except for the treatment of the nonlinearities λ' and σ'.

By introducing the general approximations $\lambda_d'(\chi_m,\chi_{m-1})$ and $\sigma_d'(\chi_m,\chi_{m-1})$ in $(\mathbf{D_Z})$, the same formulation can be used to investigate a bunch of different time–discrete schemes. A full implicit scheme corresponds to the choices $\lambda_d'(r,s)=\lambda'(r)$ and $\sigma_d'(r,s)=\sigma'(r)$. A fully explicitly treatment of nonlinearities λ' and σ' corresponds to $\lambda_d'(r,s)=\lambda'(s)$ and $\sigma_d'(r,s)=\sigma'(s)$.

The following choices for σ_d' and λ_d' fulfill **(A6)**, if **(A2)** is satisfied (see Lemma 3.1).

a) Any convex combination of $\sigma'(\chi_m)$ and $\sigma'(\chi_{m-1})$ can be used for $\sigma_d'(\chi_m,\chi_{m-1})$.

b) One particular choice for λ_d' is the following approximation for a derivative, which has been used by Niezgódka and Sprekels in [NS91, (2.3)]:

$$
\lambda_*'(r,s) := \begin{cases} \frac{\lambda(r)-\lambda(s)}{r-s}, & \text{if} \quad r\neq s,\\ \lambda'(r), & \text{if} \quad r=s. \end{cases} \tag{2.6}
$$

If one chooses this function as λ_d', the approximation for $\lambda'(\chi)\chi_t$ used in the discrete energy balance (2.4c) will coincide with the discrete differential quotient arising in the approximation of $(\lambda(\chi))_t$.

c) Assume that $\lambda\in C^2(\mathbb{R})$. If we have a uniform upper and a uniform lower bound for λ'' on $D(\beta)$, we can use every convex combination of $\lambda'(\chi_m)$ and $\lambda'(\chi_{m-1})$ for $\lambda_d'(\chi_m,\chi_{m-1})$.

If we have a uniform upper bound for λ'' on $D(\beta)$, we can use the explicit approximation $\lambda_d'(\chi_m,\chi_{m-1})=\lambda'(\chi_{m-1})$. If we have a uniform lower bound for λ'' on $D(\beta)$, we can use the implicit approximation $\lambda_d'(\chi_m,\chi_{m-1})=\lambda'(\chi_m)$.

For the time–discrete scheme there holds:

Theorem 2.1. *Assume* ***(A1)–(A6)****. Then, the scheme has a unique solution, if $|Z|$ is sufficiently small.*

Remark 2.3. We use the solution to $(\mathbf{D_Z})$ to construct an approximate solution $\left(\widehat{\theta}^Z, \widehat{u}^Z, \widehat{\chi}^Z, \overline{\xi}^Z\right)$ in $\left(L^\infty(0,T;L^2(\Omega))\right)^4$ to the Penrose–Fife system **(PF)**. The function

$\widehat{\theta}^Z$ is defined to be linear in time on $[t_{m-1}, t_m]$ for $m = 1, \dots, K$ such that $\widehat{\theta}^Z(t_k) = \theta_k$ holds for $k = 0, \dots, K$. The functions $\widehat{u}^Z$ and $\widehat{\chi}^Z$ are defined analogously. We define $\overline{\xi}^Z$ piecewise constant in time by $\overline{\xi}^Z(t) = \xi_k$ for $t \in (t_{k-1}, t_k]$ and $k = 1, \dots, K$.

The following corollary allows to check, if for a given decomposition Z the scheme has a unique solution.

Corollary 2.1. *Assume that **(A1)–(A6)** hold. There exists a solution to* $(\mathbf{D_Z})$, *if* $|Z| \le h^*$, *where* h^* *and* C_5^* *are positive constants with*

$$h^* \left(2\left(\sigma_d'(r,s)\right)^2 - C_5^*(\phi(s)+1)\right) \le c_\eta \phi(r), \quad \forall\, r, s \in D(\beta). \tag{2.7}$$

The solution to the scheme is unique, if, in addition,

$$\lambda_d'(r,s) = \lambda'(s), \quad 2\,|Z|\,|\sigma_d'(r,s) - \sigma_d'(r',s)| \le c_\eta\,|r - r'|, \quad \forall\, r, r', s \in D(\beta). \tag{2.8}$$

Remark 2.4. Assume that **(A1)–(A6)** hold. If $D(\beta)$ is bounded, Corollary 2.1 yields that the scheme has a solution. If $D(\beta)$ is unbounded, the upper bound h^* can be calculated from (2.7) for given β and σ_d'. Thanks to **(A6)**, we can always find positive h^* and C_5^*, such that (2.7) is satisfied.

If λ' is approximated explicitly and σ_d' is globally Lipschitz continuous in the first variable on $D(\beta) \times D(\beta)$, the conditions (2.8) and (2.7) lead to an computable upper bound for the time–step size to ensure the existence of a unique solution.

For σ_d' explicit, i.e. $\sigma_d'(r,s) = \sigma'(s)$, we do not get any restriction for the time–step size from (2.7) or (2.8).

2.3 Existence and approximation results

Theorem 2.2. *Assume that **(A1)–(A4)** and **(A6)** hold. Then there is a unique solution* (θ, u, χ, ξ) *to the Penrose–Fife system **(PF)**. For this solution it holds that*

$$\theta \in L^\infty(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T) \cap W^{1,\infty}(0,T;H^1(\Omega)^*), \tag{2.9}$$

$$u \in H^1(0,T;L^2(\Omega)) \cap L^\infty(\Omega_T), \tag{2.10}$$

$$\chi \in W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T). \tag{2.11}$$

As, for decompositions Z *with **(A5)**,* $|Z|$ *tends to* 0*, we have,*

$$\widehat{\theta}^Z \longrightarrow \theta \quad \text{weakly in} \quad H^1(0,T;L^2(\Omega)), \tag{2.12}$$

$$\text{weakly–star in} \quad L^\infty(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T), \tag{2.13}$$

$$\text{weakly–star in} \quad W^{1,\infty}(0,T;H^1(\Omega)^*), \tag{2.14}$$

$$\widehat{u}^Z \longrightarrow u \quad \text{weakly in} \quad H^1(0,T;L^2(\Omega)), \tag{2.15}$$

$$\text{weakly–star in} \quad L^\infty(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T), \tag{2.16}$$

$$\text{weakly in} \quad L^2(t_*,T;H^2(\Omega)),\ \forall\, 0 < t_* < T, \tag{2.17}$$

$$\widehat{\chi}^Z \longrightarrow \chi \quad \textit{weakly in} \quad H^1(0,T;H^1(\Omega)), \tag{2.18}$$

$$\textit{weakly–star in} \quad W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^2(\Omega)), \tag{2.19}$$

$$\overline{\xi}^Z \longrightarrow \xi \quad \textit{weakly–star in} \quad L^\infty(0,T;L^2(\Omega)). \tag{2.20}$$

The following error–estimate is the main result of this work.

Theorem 2.3. *Assume that* ***(A1)–(A6)*** *hold and that $|Z|$ is sufficiently small. Let (θ, u, χ, ξ) be the solution to the Penrose–Fife system* ***(PF)***.

a) If $\lambda'_d = \lambda'_*$ (cf. (2.6)), then we have a positive constant C, independent of Z, such that

$$\begin{aligned}&\left\|\widehat{\theta}^Z - \theta\right\|_{L^2(0,T;L^2(\Omega))\cap C([0,T];H^1(\Omega)^*)} + \left\|\widehat{u}^Z - u\right\|_{L^2(0,T;L^2(\Omega))}\\ &+ \left\|\widehat{\chi}^Z - \chi\right\|_{C([0,T];L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \le C\,|Z|\,.\end{aligned} \tag{2.21}$$

b) If $\lambda'_d \neq \lambda'_*$ and $\Omega \subset \mathbb{R}^2$, then (2.21) still holds.

c) If $\lambda'_d \neq \lambda'_*$ and $\Omega \subset \mathbb{R}^3$, then (2.21) holds with $|Z|$ replaced by $|Z|^{\frac{20}{23}}$.

Remark 2.5 (Numerical implementation). In a lot of physically relevant situations, see [PF90], the considered functions λ and σ are quadratic and ϕ has a quadratic lower bound, i.e. we have positive constants C_3^*, C_4^* with

$$\phi(s) + C_3^* \ge C_4^* s^2, \quad \forall\, s \in D(\beta). \tag{2.22}$$

In this situation, the scheme with

$$\sigma'_d(r,s) := \sigma'(r), \quad \lambda'_d(r,s) := \lambda'(s), \quad \forall\, r, s \in \mathbb{R}$$

is the most promising one to perform numerical computations, because of the following properties of this scheme: The coupling between the two equations (2.4c) and (2.4d) is a linear one, since $\lambda'_d(\chi_m, \chi_{m-1})$ does not depend on χ_m. Moreover, $\sigma'_d(\chi_m, \chi_{m-1})$ is linear in χ_m. Thus, a finite element discretization and a nonlinear Gauss–Seidel scheme similar to the one in [Kle97a, Sec. 10] can be used to find approximative solutions to $(\mathbf{D_Z})$. Corollary 2.1 allows us to calculate an upper bound for the time–step size to ensure the existence of a unique solution. In two space dimensions, Theorem 2.3 yields a convergence linear with respect to the time–step size, and in three dimensions the convergence in still nearly linear.

Remark 2.6. If the regularity assumption for g in **(A3)** is weakened to $g \in L^\infty(\Omega_T)$, all results of this work still holds, except the error estimates in Theorem 2.3.

3 Some properties of the approximation of the data and a special existence result

To prepare the proof of the theorems and the corollary in the last section, some notations will be fixed and some properties for the approximation of the data will be proved. Moreover,

the existence of a unique solution will be shown, under the additional condition that $D(\beta)$ is bounded.
In the sequel, we use the notation $\|\cdot\|_p$ for the $L^p(\Omega)$–norm, for all $p \in [1, \infty]$. Moreover, $\|\cdot\|_2$ will also be used for the $(L^2(\Omega))^2$ resp. $(L^2(\Omega))^3$ norm.

3.1 Properties of the data and their approximations

In the following lemma it is shown that those approximations discussed in Remark 2.2 fulfill the condition **(A6)**.

Lemma 3.1. *Assume that **(A2)** holds. Let $\omega \in [0,1]$ be given and define $\sigma_d' : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by*

$$\sigma_d'(r,s) = \omega\sigma'(r) + (1-\omega)\sigma'(s), \quad \forall\, r, s \in \mathbb{R}. \tag{3.1}$$

*a) If $\lambda_d' = \lambda_*'$ (cf. (2.6)), we have **(A6)** and*

$$\lambda_*'(r,s)(r-s) = \lambda(r) - \lambda(s), \quad \forall\, r, s \in \mathbb{R}. \tag{3.2}$$

b) Assume, in addition, $\lambda \in C^2(\mathbb{R})$ and

$$\lambda_d'(r,s) = \omega^*\lambda'(r) + (1-\omega^*)\lambda'(s), \quad \forall\, r, s \in \mathbb{R}, \tag{3.3}$$

with some $\omega^* \in [0,1]$.

*If we have positive constants C_1, C_2 such that $-C_1 \le \lambda''(s) \le C_2$ for all $s \in D(\beta)$, the assumption **(A6)** holds.*

*If $\omega^* = 0$ and we have a positive constant C_3 with $\lambda''(s) \le C_3$ for all $s \in D(\beta)$, the assumption **(A6)** is satisfied.*

*If $\omega^* = 1$ and we have a positive constant C_4 with $-C_4 \le \lambda''(s)$ for all $s \in D(\beta)$, the assumption **(A6)** holds.*

Proof. First, we consider part (a) of the lemma. Thanks to (2.6), we have (3.2) and $\lambda_*'(r,s) = \int_0^1 \lambda'(s + \tau(r-s))\,\mathrm{d}\tau$. Hence, for $\lambda_d' = \lambda_*'$, we can use (3.1), Schwarz's inequality, and **(A2)**, to show that **(A6)** is satisfied. This yields part (a) of the Lemma.
To prove part (b) of the lemma, we need only to show that the last estimate in **(A6)**, i.e. (2.5), is satisfied, since the remaining assumptions in **(A6)** follow by an argumentation similar to the one above. For $r, s \in D(\beta)$, applying Taylor's formula and (3.3) gives $\tau, \mu \in D(\beta)$ such that

$$-\lambda_d'(r,s)(r-s) + \lambda(r) - \lambda(s) = \frac{1}{2}\left(-\omega^*\lambda''(\tau) + (1-\omega^*)\lambda''(\mu)\right)(r-s)^2.$$

Now, we see immediately that (2.5) holds under the considered assumptions. $\square$

Lemma 3.2. *Assume that **(A3)** holds. Then there exist positive constants $C_1, C_2, \ldots, C_6$, such that, for all decompositions Z with **(A5)**, the functions g_m, γ_m, and ζ_m defined in* (2.3) *fulfill, for $1 \le m \le K$,*

$$C_1 \|v\|_{H^1(\Omega)}^2 \le \kappa \|\nabla v\|_2^2 + \int\limits_\Gamma \gamma_m v^2 \,\mathrm{d}\sigma \le C_2 \|v\|_{H^1(\Omega)}^2, \quad \forall v \in H^1(\Omega),$$

$$\gamma_m v \in H^{\frac{1}{2}}(\Gamma), \qquad \|\gamma_m v\|_{H^{\frac{1}{2}}(\Gamma)} \le C_3 \|v\|_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega),$$

$$c_\zeta \le \zeta_m \quad a.e.\ in \quad \Gamma, \qquad \left|\int\limits_\Gamma \zeta_m v \,\mathrm{d}\sigma\right| + \left|\int\limits_\Omega g_m v \,\mathrm{d}x\right| \le C_4 \|v\|_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega),$$

$$\|g_m\|_\infty + \|\gamma_m\|_{C^1(\Gamma)} + \|\zeta_m\|_{L^\infty(\Gamma)} + \|\zeta_m\|_{H^{\frac{1}{2}}(\Gamma)} \le C_5,$$

and

$$\max_{1 \le m \le K-1} \left\| \frac{\gamma_{m+1} - \gamma_m}{h_m} \right\|_{L^\infty(\Gamma)} + \sum_{m=1}^{K-1} h_m \left\| \frac{\zeta_{m+1} - \zeta_m}{h_m} \right\|_{L^2(\Gamma)}^2 \le C_6,$$

*where the positive constants c_ζ, c_γ are specified in **(A3)**.*

Proof. This lemma follows from **(A3)**, **(A5)**, the trace–mapping from $H^1(\Omega)$ to $H^{\frac{1}{2}}(\Gamma)$, and the interpolation of $H^{\frac{1}{2}}(\Gamma)$ by $H^1(\Gamma)$ and $L^2(\Gamma)$. □

3.2 The existence proof for $D(\beta)$ bounded

Lemma 3.3. *Assume that **(A1)–(A6)** hold and that $D(\beta)$ is bounded. Then there exists a solution to* $(\mathbf{D_Z})$.

Proof. From (2.4f), we get $\theta_0, u_0, \chi_0, \xi_0$. Now, we assume that $\theta_{m-1} \in L^2(\Omega), \chi_{m-1} \in H^2(\Omega)$ for some $m \in \{1, \ldots, K\}$ are given. To show that there exists a solution to the system in $(\mathbf{D_Z})$, i.e. to (2.4a)–(2.4e), we will first consider the discrete energy balance equation and the discrete equation for the order parameter separately. Afterwards, we will rewrite the system as a fixed point problem and apply Schauder's fixed point theorem.

Lemma 3.4. *For every $\chi \in L^\infty(\Omega)$, there is a unique $\tilde{u} \in H^2(\Omega)$ such that*

$$0 < \tilde{u} \quad a.e.\ in \quad \Omega, \qquad \frac{1}{\tilde{u}} \in L^2(\Omega), \quad -\kappa \frac{\partial \tilde{u}}{\partial n} = \gamma_m \tilde{u} - \zeta_m \quad a.e.\ in \quad \Gamma, \tag{3.4}$$

$$-\frac{c_0}{\tilde{u}} - h_m \kappa \Delta \tilde{u} = -c_0 \theta_{m-1} - h_m g_m + \lambda_d'(\chi, \chi_{m-1})(\chi - \chi_{m-1}) \quad a.e.\ in \quad \Omega. \tag{3.5}$$

Proof. Let $\chi \in L^\infty(\Omega)$ be given. Thanks to **(A6)** and $\chi_{m-1} \in C(\bar{\Omega})$, we have

$$\lambda_d'(\chi, \chi_{m-1})(\chi - \chi_{m-1}) \in L^2(\Omega).$$

By translating the proof of [Bré71, Corollary 13], we see that the operator corresponding to (3.4) and the left–hand side of (3.5) is maximal monotone. By showing that this operator is also coercive, we obtain that the operator is also surjective. The injectivity follows by estimating the difference between two given solutions. Details can be found in [Kle97a, Lemma 5.1]. □

Lemma 3.5. *For every $\chi \in L^\infty(\Omega)$, $\tilde{u} \in L^2(\Omega)$ there exists a unique $\tilde{\chi}$ such that*

$$\tilde{\chi} \in H^2(\Omega), \qquad \tilde{\chi} \in D(\beta) \quad a.e. \ in \quad \Omega, \qquad \frac{\partial \tilde{\chi}}{\partial n} = 0 \quad a.e. \ in \quad \Gamma, \tag{3.6}$$

$$\eta \frac{\tilde{\chi} - \chi_{m-1}}{h_m} - \varepsilon \Delta \tilde{\chi} + \beta(\tilde{\chi}) \ni \sigma_d'(\chi, \chi_{m-1}) - \lambda_d'(\chi, \chi_{m-1}) \tilde{u} \quad a.e. \ in \quad \Omega, \tag{3.7}$$

$$-\eta \frac{\tilde{\chi} - \chi_{m-1}}{h_m} + \varepsilon \Delta \tilde{\chi} + \sigma_d'(\chi, \chi_{m-1}) - \lambda_d'(\chi, \chi_{m-1}) \tilde{u} \ \in \ L^2(\Omega). \tag{3.8}$$

Proof. By **(A1)** and **(A3)**, we can rewrite (3.6)–(3.8) as

$$\frac{c_\eta}{h_m} \tilde{\chi} + B \tilde{\chi} \ni \sigma_d'(\chi, \chi_{m-1}) - \lambda_d'(\chi, \chi_{m-1}) \tilde{u} + \frac{\eta}{h_m} \chi_{m-1}, \tag{3.9}$$

where $B : L^2(\Omega) \to \{W \subseteq L^2(\Omega)\}$ is a nonlinear operator. Using [Bré71, Corollary 13], we see that this operator is maximal monotone. Details can be found in [Kle97a, (5.7)–(5.8) and Lemma 5.5].
Because of **(A6)**, **(A3)**, $\chi \in L^\infty(\Omega)$, $\chi_{m-1} \in H^2(\Omega) \subset C(\overline{\Omega})$, we see that the right–hand side of (3.9) is in $L^2(\Omega)$. Hence, [Bré71, Theorem 2] yields that there is a unique solution $\tilde{\chi}$ to (3.6)–(3.8). ☐

In this proof, C_i, for $i \in \mathbb{N}$, will always denote generic positive constants, independent of $\chi \in \mathcal{M}$ with

$$\mathcal{M} := \left\{ \chi \in L^2(\Omega) : \chi \in \overline{D(\beta)} \quad \text{a.e. in} \quad \Omega \right\}. \tag{3.10}$$

This is a closed and convex set.
We have

Lemma 3.6. *The functions $\sigma_d'(\cdot, \chi_{m-1})$ and $\lambda_d'(\cdot, \chi_{m-1})$ are Lipschitz continuous on $\overline{D(\beta)}$ and there is a positive constant C_1 such that, for all $\chi \in \mathcal{M}$,*

$$\|\lambda_d'(\chi, \chi_{m-1})\|_\infty + \|\sigma_d'(\chi, \chi_{m-1})\|_\infty + \|\chi\|_\infty + \|\chi_{m-1}\|_\infty \le C_1. \tag{3.11}$$

Proof. Since $\overline{D(\beta)}$ is bounded and $\chi_{m-1} \in H^2(\Omega) \subset C(\overline{\Omega})$, **(A6)** yields that the assertions of this lemma hold. ☐

Combining Lemma 3.4 and Lemma 3.5, we see that for every $\chi \in \mathcal{M}$ there is a unique $\tilde{u} \in H^2(\Omega)$ and a unique $\Psi(\chi) := \tilde{\chi} \in H^2(\Omega)$ such that (3.4)–(3.5) and (3.6)–(3.8) hold.
This defines a mapping $\Psi : \mathcal{M} \to \mathcal{M}$ and any fixed point of Ψ leads to a solution to the system in $(\mathbf{D_Z})$, i.e. to (2.4a)–(2.4e). Therefore, it is sufficient to prove that Ψ has a fixed point.
We test (3.5) by $h_m \tilde{u}$, apply Green's formula, Lemma 3.2, Hölder's inequality, (3.4), (3.11), and Young's inequality to conclude that

$$\begin{aligned} & C_2 \|\tilde{u}\|^2_{H^1(\Omega)} \\ \le & c_0 |\Omega| + h_m \int_\Gamma \zeta_m \tilde{u} \, \mathrm{d}\sigma + \int_\Omega \left(-c_0 \theta_{m-1} - h_m g_m + \lambda_d'(\chi, \chi_{m-1}) (\chi - \chi_{m-1})\right) \tilde{u} \, \mathrm{d}x \\ \le & C_3 + \frac{C_2}{2} \|\tilde{u}\|^2_{H^1(\Omega)}. \end{aligned} \tag{3.12}$$

Owing to **(A1)**, we have $ws \geq 0$ for all $s \in D(\beta)$, $w \in \beta(s)$. Therefore, by testing (3.7) by $\tilde{\chi}$ and applying **(A3)**, Green's formula, (3.6), (3.11), Hölder's inequality, (3.12), and Young's inequality, we get

$$C_4 \|\tilde{\chi}\|^2_{H^1(\Omega)} \leq \left\| \eta \frac{\chi_{m-1}}{h_m} + \sigma_d'(\chi, \chi_{m-1}) - \lambda'(\chi, \chi_{m-1})\tilde{u} \right\|_2 \|\tilde{\chi}\|_2 \leq C_5 + \frac{C_4}{2} \|\tilde{\chi}\|_2^2 .$$

Hence, we see that $\tilde{\chi} \in \mathcal{M}_1$ with

$$\mathcal{M}_1 := \left\{ \bar{\chi} \in \mathcal{M} : \|\bar{\chi}\|^2_{H^1(\Omega)} \leq 2\frac{C_5}{C_4} \right\}.$$

Therefore, we observe that $\mathcal{M}_1$ is a nonempty, convex, compact set in $L^2(\Omega)$ and, by construction, that Ψ maps $\mathcal{M}_1$ into itself. Thanks to Lemma 3.7, Ψ is on $\mathcal{M}_1$ continuous. Now, Schauder's fixed point theorem yields the existence of a fixed point of Ψ in $\mathcal{M}_1$. □

Lemma 3.7. *$\Psi : \mathcal{M} \to \mathcal{M}$ is $L^2(\Omega)$–continuous.*

Proof. Let χ_1^*, χ_2^* in $\mathcal{M}$ be arbitrary, and

$$\tilde{\chi}_1 := \Psi(\chi_1^*), \quad \tilde{\chi}_2 := \Psi(\chi_2^*), \quad \chi^* := \chi_1^* - \chi_2^*, \quad \tilde{\chi} := \tilde{\chi}_1 - \tilde{\chi}_2.$$

Combining (3.4)–(3.5), (3.6)–(3.8), and the definition of Ψ, we find $\tilde{u}_1, \tilde{u}_2 \in H^2(\Omega)$, $\tilde{\xi}_1, \tilde{\xi}_2 \in L^2(\Omega)$ such that

$$\tilde{u}_1 > 0, \quad \tilde{u}_2 > 0, \quad \tilde{\xi}_1 \in \beta(\tilde{\chi}_1), \quad \tilde{\xi}_2 \in \beta(\tilde{\chi}_2) \quad \text{a.e. in} \quad \Omega, \tag{3.13}$$

$$\begin{aligned} &-c_0\left(\frac{1}{\tilde{u}_1} - \frac{1}{\tilde{u}_2}\right) - h_m\kappa\Delta\left(\tilde{u}_1 - \tilde{u}_2\right) \\ &= \lambda_d'(\chi_1^*, \chi_{m-1})(\chi_1^* - \chi_{m-1}) - \lambda_d'(\chi_2^*, \chi_{m-1})(\chi_2^* - \chi_{m-1}) \quad \text{a.e. in} \quad \Omega, \end{aligned} \tag{3.14}$$

$$\begin{aligned} &\eta\frac{\tilde{\chi}}{h_m} - \varepsilon\Delta\tilde{\chi} + \tilde{\xi}_1 - \tilde{\xi}_2 = -\lambda_d'(\chi_1^*, \chi_{m-1})\tilde{u}_1 + \lambda_d'(\chi_2^*, \chi_{m-1})\tilde{u}_2 \\ &\qquad + \sigma_d'(\chi_1^*, \chi_{m-1}) - \sigma_d'(\chi_2^*, \chi_{m-1}) \quad \text{a.e. in} \quad \Omega, \end{aligned} \tag{3.15}$$

$$-\kappa\frac{\partial\left(\tilde{u}_1 - \tilde{u}_2\right)}{\partial n} = \gamma_m\left(\tilde{u}_1 - \tilde{u}_2\right), \quad \frac{\partial\tilde{\chi}}{\partial n} = 0 \qquad \text{a.e. in} \quad \Gamma. \tag{3.16}$$

Testing (3.14) by $\tilde{u} := \tilde{u}_1 - \tilde{u}_2$, integrating by parts, and using (3.16), (3.13), Lemma 3.2, Hölder's inequality, Lemma 3.6, and Young's inequality, we deduce

$$\begin{aligned} &c_0\left\|\frac{\tilde{u}}{\sqrt{\tilde{u}_1\tilde{u}_2}}\right\|_2^2 + C_6\|\tilde{u}\|^2_{H^1(\Omega)} \\ &\leq \int_\Omega \left(\lambda_d'(\chi_1^*, \chi_{m-1})\chi^* + \left(\lambda_d'(\chi_1^*, \chi_{m-1}) - \lambda_d'(\chi_2^*, \chi_{m-1})\right)(\chi_2^* - \chi_{m-1})\right)\tilde{u}\,\mathrm{d}x \\ &\leq C_7\|\chi^*\|_2\|\tilde{u}\|_2 \leq C_8\|\chi^*\|_2^2 + \frac{C_6}{2}\|\tilde{u}\|_2^2 . \end{aligned} \tag{3.17}$$

We test (3.15) by $\tilde{\chi}$ and use (3.13), the monotonicity of β, **(A3)**, (3.16), and the generalized Hölder's inequality (see Lemma AP.2) to derive

$$C_9 \|\tilde{\chi}\|^2_{H^1(\Omega)} \le \|\lambda_d'(\chi_1^*, \chi_{m-1})\|_{\frac{3}{2}} \|\tilde{u}\|_6 \|\tilde{\chi}\|_6 + \|\lambda_d'(\chi_1^*, \chi_{m-1}) - \lambda_d'(\chi_2^*, \chi_{m-1})\|_{\frac{3}{2}} \|\tilde{u}_2\|_6 \|\tilde{\chi}\|_6$$
$$+ \|\sigma_d'(\chi_1^*, \chi_{m-1}) - \sigma_d'(\chi_2^*, \chi_{m-1})\|_2 \|\tilde{\chi}\|_2 .$$

Because of Lemma 3.6, (AP.1), (3.17), and (3.12), we see

$$C_9 \|\tilde{\chi}\|^2_{H^1(\Omega)} \le C_{10} \|\chi^*\|_2 \|\tilde{\chi}\|_{H^1(\Omega)} .$$

Hence, thanks to Young's inequality, we have shown that Ψ is $L^2(\Omega)$–continuous. □

4 Uniform estimates

In this section, uniform estimates for the solutions to the time–discrete scheme are derived. Assume that **(A1)**–**(A6)** hold and that $|Z| \le h^*$, where h^* and C_5^* are positive constants such that (2.7) is satisfied.
Let $\beta^* := \partial\phi^*$ and $\phi^* : \mathbb{R} \to [0, \infty]$ be either ϕ or the function defined by

$$\phi^*(s) = \begin{cases} \phi(s), & \text{if } |s| \le B, \\ \infty, & \text{otherwise,} \end{cases} \tag{4.1}$$

for some $B > \|\chi^0\|_\infty$. In the light of **(A1)**, we see that ϕ^* is a convex, lower semicontinuous function with

$$0 \le \phi \le \phi^* \text{ on } \mathbb{R}, \quad 0 \in D(\beta^*), \quad \mathrm{int}D(\beta^*) \ne \emptyset, \quad 0 \in \beta^*(0), \quad \phi^*|_{D(\beta^*)} = \phi|_{D(\beta^*)} . \tag{4.2}$$

Now, a modified version of the time–discrete scheme is considered, where β in $(\mathbf{D_Z})$, i.e. in (2.4b), is replaced by β^*. Let any solution to this scheme be given.
In the sequel, C_i, for $i \in \mathbb{N}$, will always denote positive generic constants, independent of the decomposition Z, the considered choice of ϕ^*, and the solution itself.
Remark 4.1.

Recalling (2.4a), (2.4b), (2.4e), (2.4f), **(A4)**, and the definition of ϕ^*, we see that

$$\begin{aligned} &0 < u_m = \frac{1}{\theta_m}, \quad \chi_m \in D(\beta^*) \subseteq D(\beta), \quad \xi_m \in \beta^*(\chi_m) = \partial\phi^*(\chi_m) \text{ a.e. in } \Omega, \\ &\chi_m \in H^2(\Omega), \quad \frac{\partial \chi_m}{\partial n} = 0 \quad \text{a.e. in} \quad \Gamma, \quad \forall\, 0 \le m \le K. \end{aligned} \tag{4.3}$$

Applying (2.4c), Green's formula, and (2.4e), we deduce that

$$\begin{aligned} &\int_\Omega \left(c_0 \frac{\theta_m - \theta_{m-1}}{h_m} + \frac{\lambda_m - \lambda_{m-1}}{h_m} \right) v \,\mathrm{d}x \, - \kappa \int_\Omega \nabla u_m \bullet \nabla v \,\mathrm{d}x \\ &- \int_\Gamma \gamma_m u_m v \,\mathrm{d}\sigma \, = \int_\Omega g_m v \,\mathrm{d}x \, - \int_\Gamma \zeta_m v \,\mathrm{d}\sigma \,, \quad \forall\, v \in H^1(\Omega), \quad 1 \le m \le K, \end{aligned} \tag{4.4}$$

with

$$\lambda_0 := \lambda(\chi_0), \;\; \lambda_m := \lambda_{m-1} + \lambda_d'(\chi_m, \chi_{m-1})(\chi_m - \chi_{m-1}) \text{ a.e. in } \Omega, \; \forall\, 1 \le m \le K. \tag{4.5}$$

The following Lemmas use ideas from [HSZ96, SZ93, CS97, Hor93, HS, Lau93, Lau94, Kle97a]

Lemma 4.1. *a) There is a positive constant C_1 such that*

$$\|\phi^*(\chi_0)\|_1 + \|\xi_0\|_2 + \|\lambda(\chi_0)\|_1 + \|\lambda_0\|_6 + \|\lambda_d'(\chi_0,\chi_0)\|_2 + \|\sigma_d'(\chi_0,\chi_0)\|_2 + \|\chi_0\|_{H^2(\Omega)} + \|\theta_0\|_{H^1(\Omega)\cap L^\infty(\Omega)} + \|u_0\|_{H^1(\Omega)\cap L^\infty(\Omega)} + \|\ln(\theta_0)\|_1 \le C_1. \tag{4.6}$$

b) Let $\chi_{-1} \in L^2(\Omega)$ be defined by

$$\eta \frac{\chi_0 - \chi_{-1}}{h_0} - \varepsilon\Delta\chi_0 + \xi_0 + \sigma_d'(\chi_0,\chi_0) = -\lambda_d'(\chi_0,\chi_0)u_0 \quad a.e.\ in \quad \Omega, \tag{4.7}$$

with $h_0 := |Z|$. We have a positive constant C_2 such that

$$\left\| \sqrt{\eta}\ \left(\frac{\chi_0 - \chi_{-1}}{h_0}\right)\right\|_2^2 \le C_2. \tag{4.8}$$

Proof. If $\phi^* = \phi$, we use the initial condition (2.4f), **(A2)**, **(A4)**, Sobolev's embedding Theorem, **(A6)**, and (4.5) to show that (4.6) is satisfied. If $\phi^* \ne \phi$, in addition, (4.1) and $B > \|\chi^0\|_\infty$ are applied. Combining (4.7), (4.6), and **(A3)** leads to (4.8). □

Lemma 4.2. *There are two positive constants C_3, C_4 such that*

$$\max_{0\le m\le K}\left(\|\theta_m\|_1 + \|\ln(\theta_m)\|_1 + \|\chi_m\|_{H^1(\Omega)}^2 + \|\phi(\chi_m)\|_1\right) + \sum_{m=1}^K h_m\|u_m\|_{H^1(\Omega)}^2 + \sum_{m=1}^K h_m\left\|\frac{\chi_m - \chi_{m-1}}{h_m}\right\|_2^2 + \sum_{m=1}^K \|\chi_m - \chi_{m-1}\|_{H^1(\Omega)}^2 \le C_3, \tag{4.9}$$

$$\max_{1\le m\le K}\|\sigma_d'(\chi_m,\chi_{m-1})\|_2 \le C_4. \tag{4.10}$$

Proof. Testing (2.4d) by $(\chi_m - \chi_{m-1})$, taking the sum from $m = 1$ to $m = k$, and using **(A3)**, Green's formula, (4.3), (AP.5), (4.6), (4.2), (4.5), Schwarz's inequality, and Young's inequality, we deduce

$$\frac{c_\eta}{2}\sum_{m=1}^k h_m\left\|\frac{\chi_m - \chi_{m-1}}{h_m}\right\|_2^2 + \frac{\varepsilon}{2}\|\nabla\chi_k\|_2^2 + \frac{\varepsilon}{2}\sum_{m=1}^k\|\nabla\chi_m - \nabla\chi_{m-1}\|_2^2 + \|\phi(\chi_k)\|_1 \le C_5 - \sum_{m=1}^k\int_\Omega(\lambda_m - \lambda_{m-1})\,u_m\,\mathrm{d}x + \frac{1}{2c_\eta}\sum_{m=1}^k h_m\|\sigma_d'(\chi_m,\chi_{m-1})\|_2^2. \tag{4.11}$$

Let $\alpha := \min\left(\frac{1}{2C_1^*}, \frac{c_\eta}{6C_2^*T}\right)$, with C_1^*, C_2^* as in **(A2)** and **(A6)**. For $1 \le m \le K$, we insert $v = h_m\alpha - h_m u_m$ in (4.4), use (4.3) and that $\frac{-1}{s}$ is the derivative of the convex function $-\ln(s)$, take the sum from $m = 1$ to $m = k$, and apply Lemma 3.2, (4.6), and Young's inequality, to show that

$$c_0\int_\Omega(-\ln(\theta_k))\,\mathrm{d}x + \alpha c_0\|\theta_k\|_1 + C_6\sum_{m=1}^k h_m\|u_m\|_{H^1(\Omega)}^2 \le C_7 + \sum_{m=1}^k\int_\Omega(\lambda_m - \lambda_{m-1})(u_m - \alpha)\,\mathrm{d}x\,. \tag{4.12}$$

Because of (4.5), **(A6)**, **(A2)**, (4.6), Young's inequality, and the definition of α, we have

$$-\alpha \sum_{m=1}^{k} \int_{\Omega} (\lambda_m - \lambda_{m-1})\, \mathrm{d}x \le C_8 + \frac{1}{2} \int_{\Omega} \phi(\chi_k)\, \mathrm{d}x + \frac{c_\eta}{6} \sum_{m=1}^{k} h_m \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_2^2 .$$

Hence, by using Lemma AP.8 and adding (4.12) to (4.11), we deduce

$$\begin{aligned} &C_9 \|\theta_k\|_1 + c_0 \|\ln(\theta_k)\|_1 + C_6 \sum_{m=1}^{k} h_m \|u_m\|_{H^1(\Omega)}^2 + \frac{c_\eta}{3} \sum_{m=1}^{k} h_m \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_2^2 + \frac{\varepsilon}{2} \|\nabla \chi_k\|_2^2 \\ &+ \frac{\varepsilon}{2} \sum_{m=1}^{k} \|\nabla (\chi_m - \chi_{m-1})\|_2^2 + \frac{1}{2} \|\phi(\chi_k)\|_1 \le C_{10} + \frac{1}{2c_\eta} \sum_{m=1}^{k} h_m \|\sigma_d'(\chi_m, \chi_{m-1})\|_2^2 . \qquad (4.13) \end{aligned}$$

Since **(A6)**, (2.7), and $|Z| \le h^*$ yield

$$\|\sigma_d'(\chi_m, \chi_{m-1})\|_2^2 \le C_2^* \left(\|\phi(\chi_m)\|_1 + \|\phi(\chi_{m-1})\|_1 + \int_{\Omega} 1\, \mathrm{d}x \right), \quad \forall\, 1 \le m \le K, \qquad (4.14)$$

$$\frac{h_k}{2c_\eta} \|\sigma_d'(\chi_k, \chi_{k-1})\|_2^2 \le \frac{1}{4} \|\phi(\chi_k)\|_1 + C_{11} \left(h_k \|\chi_{k-1}\|_1 + 1 \right),$$

we obtain from (4.13), **(A5)**, the discrete version of Gronwall's lemma, and (4.6) that (4.9) is satisfied. Therefore, (4.10) holds because of (4.14). □

Lemma 4.3. *There exists a constant C_{12} such that*

$$\begin{aligned} &\max_{0 \le m \le K} \left(\left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_2^2 + \|u_m\|_{H^1(\Omega)}^2 \right) + \sum_{m=1}^{K} h_m \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_{H^1(\Omega)}^2 \\ &+ \sum_{m=1}^{K} \left\| (\xi_m - \xi_{m-1}) \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_1 + \sum_{m=1}^{K} \left\| \frac{\chi_m - \chi_{m-1}}{h_m} - \frac{\chi_{m-1} - \chi_{m-2}}{h_{m-1}} \right\|_2^2 \\ &+ \sum_{m=1}^{K} h_m \left\| \frac{u_m - u_{m-1}}{h_m \sqrt{u_m u_{m-1}}} \right\|_2^2 + \sum_{m=1}^{K} \|u_m - u_{m-1}\|_{H^1(\Omega)}^2 \le C_{12}, \qquad (4.15) \end{aligned}$$

with χ_{-1}, h_0 as in Lemma 4.1.

Proof. Inserting $v = -(u_m - u_{m-1})$ in (4.4), taking the sum from $m = 1$ to $m = k$, and applying (4.3), (AP.5), (AP.4), Lemma 3.2, (4.9), (4.6), the generalized Hölder's inequality, $h_m \le 2h_{m-1}$, and Young's inequality, we deduce that

$$\begin{aligned} &\frac{c_o}{2} \sum_{m=1}^{k} h_m \left\| \frac{u_m - u_{m-1}}{h_m \sqrt{u_m u_{m-1}}} \right\|_2^2 + C_{13} \|u_k\|_{H^1(\Omega)}^2 + C_{13} \sum_{m=1}^{k} \|u_m - u_{m-1}\|_{H^1(\Omega)}^2 \\ \le &C_{14} + \sum_{m=1}^{k} \int_{\Omega} \frac{\lambda_m - \lambda_{m-1}}{h_m} (u_m - u_{m-1})\, \mathrm{d}x . \qquad (4.16) \end{aligned}$$

For $2 \le m \le K$, we test the difference of (2.4d) for m and $m-1$ by $\frac{\chi_m - \chi_{m-1}}{h_m}$. By applying **(A3)**, Green's formula, (4.3), the monotonicity of β, (4.5), and (AP.5), we obtain that

$$
\begin{aligned}
&\frac{1}{2}\left\|\sqrt{\eta}\,\frac{\chi_m - \chi_{m-1}}{h_m}\right\|_2^2 - \frac{1}{2}\left\|\sqrt{\eta}\,\frac{\chi_{m-1} - \chi_{m-2}}{h_{m-1}}\right\|_2^2 + \frac{c_\eta}{2}\left\|\frac{\chi_m - \chi_{m-1}}{h_m} - \frac{\chi_{m-1} - \chi_{m-2}}{h_{m-1}}\right\|_2^2 \\
&\quad + \varepsilon h_m \left\|\nabla\left(\frac{\chi_m - \chi_{m-1}}{h_m}\right)\right\|_2^2 + \left\|(\xi_m - \xi_{m-1})\frac{\chi_m - \chi_{m-1}}{h_m}\right\|_1 \\
&\le -\int_\Omega \left(\frac{\lambda_m - \lambda_{m-1}}{h_m} u_m - \lambda'_{d,m-1}\frac{\chi_m - \chi_{m-1}}{h_m} u_{m-1}\right) \mathrm{d}x \\
&\quad + \int_\Omega \left(\sigma'_d(\chi_m, \chi_{m-1}) - \sigma'_{d,m-1}\right) \frac{\chi_m - \chi_{m-1}}{h_m}\,\mathrm{d}x\,,
\end{aligned}
\tag{4.17}
$$

$$
\text{with}\quad \lambda'_{d,m-1} := \lambda'_d(\chi_{m-1}, \chi_{m-2}), \quad \sigma'_{d,m-1} := \sigma'_d(\chi_{m-1}, \chi_{m-2}) \quad \text{a.e. in} \quad \Omega. \tag{4.18}
$$

Testing the difference of (2.4d) for $m = 1$ and (4.7) by $\frac{\chi_1 - \chi_0}{h_1}$ and using the same argumentation as above, we deduce that (4.17) holds also for $m = 1$ with

$$
\lambda'_{d,0} := \lambda'_d(\chi_0, \chi_0), \quad \sigma'_{d,0} := \sigma'_d(\chi_0, \chi_0) \quad \text{a.e. in} \quad \Omega. \tag{4.19}
$$

Summing up (4.17) from $m = 1$ to $m = k$, adding the resulting estimate to (4.16), and using **(A3)**, (4.9), and (4.8), we conclude that

$$
\begin{aligned}
&\frac{c_\eta}{2}\left\|\frac{\chi_k - \chi_{k-1}}{h_k}\right\|_2^2 + \frac{c_\eta}{2}\sum_{m=1}^{k}\left\|\frac{\chi_m - \chi_{m-1}}{h_m} - \frac{\chi_{m-1} - \chi_{m-2}}{h_{m-1}}\right\|_2^2 \\
&+ C_{15}\sum_{m=1}^{k} h_m \left\|\frac{\chi_m - \chi_{m-1}}{h_m}\right\|_{H^1(\Omega)}^2 + \sum_{m=1}^{k}\left\|(\xi_m - \xi_{m-1})\frac{\chi_m - \chi_{m-1}}{h_m}\right\|_1 \\
&+ \frac{c_o}{2}\sum_{m=1}^{k} h_m \left\|\frac{u_m - u_{m-1}}{h_m\sqrt{u_m u_{m-1}}}\right\|_2^2 + C_{13}\|u_k\|_{H^1(\Omega)}^2 + C_{13}\sum_{m=1}^{k}\|u_m - u_{m-1}\|_{H^1(\Omega)}^2 \\
\le{}& C_{16} + I_{1,k} + I_{2,k},
\end{aligned}
\tag{4.20}
$$

$$
\text{with}\quad I_{1,k} := \sum_{m=1}^{k}\int_\Omega \left(\lambda'_{d,m-1}\frac{\chi_m - \chi_{m-1}}{h_m} - \frac{\lambda_m - \lambda_{m-1}}{h_m}\right) u_{m-1}\,\mathrm{d}x\,, \tag{4.21}
$$

$$
I_{2,k} := \sum_{m=1}^{k}\int_\Omega \left(\sigma'_d(\chi_m, \chi_{m-1}) - \sigma'_{d,m-1}\right)\frac{\chi_m - \chi_{m-1}}{h_m}\,\mathrm{d}x\,. \tag{4.22}
$$

Using (4.5), the generalized Hölder's inequality, and Schwarz's inequality, we deduce that

$$
I_{1,k} \le \left(\max_{1\le m\le k}\left\|\frac{\chi_m - \chi_{m-1}}{h_m}\right\|_2\right)\sqrt{I_{3,k}}\,\sqrt{\sum_{m=1}^{k} h_{m-1}\|u_{m-1}\|_6^2},
$$

$$
\text{with } I_{3,k} := \sum_{m=1}^{k}\frac{1}{h_{m-1}}\left\|\lambda'_{d,m-1} - \lambda'_d(\chi_m, \chi_{m-1})\right\|_3^2. \tag{4.23}
$$

Now, owing to (AP.1), (4.9), (4.6), and Young's inequality, we observe that

$$I_{1,k} = \frac{c_\eta}{4} \max_{1\le m\le k} \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_2^2 + C_{17} I_{3,k}. \tag{4.24}$$

Since $\frac{1}{3} = \frac{1}{p_1} + \frac{p}{6}$ holds for $p_1 := \frac{6}{2-p}$, we obtain, by (4.18), (4.19), **(A6)**, the generalized Hölder's inequality, $h_m \le 2h_{m-1}$, (AP.1), and (4.9), that

$$\begin{aligned} I_{3,k} \le & C_{18} \sum_{m=2}^{k} \left(\frac{h_m^2}{h_{m-1}} \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_{p_1}^2 + h_{m-1} \left\| \frac{\chi_{m-1} - \chi_{m-2}}{h_{m-1}} \right\|_{p_1}^2 \right) \\ & \left(\|\chi_m^p\|_{\frac{6}{p}} + \|\chi_{m-1}^p\|_{\frac{6}{p}} + \|\chi_{m-2}^p\|_{\frac{6}{p}} + 1 \right)^2 \\ & + C_{19} \frac{h_1^2}{|Z|} \left\| \frac{\chi_1 - \chi_0}{h_1} \right\|_{p_1}^2 \left(\|\chi_1^p\|_{\frac{6}{p}} + \|\chi_0^p\|_{\frac{6}{p}} + 1 \right)^2 \le C_{20} \sum_{m=1}^{k} h_m \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_{\frac{6}{2-p}}^2 . \end{aligned}$$

Because of $p < 1$, we can use the Gagliardo–Nirenberg inequality (see Lemma AP.5) and Young's inequality to deduce

$$C_{17} I_{3,k} \le \frac{C_{15}}{4} \sum_{m=1}^{k} h_m \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_{H^1(\Omega)}^2 + C_{21} \sum_{m=1}^{k} h_m \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_2^2. \tag{4.25}$$

Defining $q_1 := \frac{12}{6-q}$, we have $1 = \frac{1}{q_1} + \frac{q}{6} + \frac{1}{q_1}$. It follows from (4.22), (4.18), (4.19), **(A6)**, and the generalized Hölder's inequality that

$$\begin{aligned} I_{2,k} \le & C_{22} \sum_{m=2}^{k} \left(h_m \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_{q_1} + h_{m-1} \left\| \frac{\chi_{m-1} - \chi_{m-2}}{h_{m-1}} \right\|_{q_1} \right) \\ & \left(\|\chi_m^q\|_{\frac{6}{q}} + \|\chi_{m-1}^q\|_{\frac{6}{q}} + \|\chi_{m-2}^q\|_{\frac{6}{q}} + 1 \right) \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_{q_1} \\ & + C_{23} h_1 \left\| \frac{\chi_1 - \chi_0}{h_1} \right\|_{q_1} \left(\|\chi_1^q\|_{\frac{6}{q}} + \|\chi_0^q\|_{\frac{6}{q}} + 1 \right) \left\| \frac{\chi_1 - \chi_0}{h_1} \right\|_{q_1} . \end{aligned}$$

Using (AP.1), (4.9), Young's inequality, **(A5)**, the Gagliardo–Nirenberg inequality, and $q < 4$, we obtain that

$$I_{2,k} \le \frac{C_{15}}{4} \sum_{m=1}^{k} h_m \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_{H^1(\Omega)}^2 + C_{24} \sum_{m=1}^{k} h_m \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_2^2. \tag{4.26}$$

Combining (4.20), (4.24)–(4.26), and (4.9), we conclude that

$$
\begin{aligned}
&\frac{c_\eta}{2}\left\|\frac{\chi_k-\chi_{k-1}}{h_k}\right\|_2^2+\frac{c_\eta}{2}\sum_{m=1}^{k}\left\|\frac{\chi_m-\chi_{m-1}}{h_m}-\frac{\chi_{m-1}-\chi_{m-2}}{h_{m-1}}\right\|_2^2\\
&+\frac{C_{15}}{2}\sum_{m=1}^{k}h_m\left\|\frac{\chi_m-\chi_{m-1}}{h_m}\right\|_{H^1(\Omega)}^2+\sum_{m=1}^{k}\left\|(\xi_m-\xi_{m-1})\frac{\chi_m-\chi_{m-1}}{h_m}\right\|_1\\
&+\frac{c_o}{2}\sum_{m=1}^{k}h_m\left\|\frac{u_m-u_{m-1}}{h_m\sqrt{u_m u_{m-1}}}\right\|_2^2+C_{13}\left\|u_k\right\|_{H^1(\Omega)}^2+C_{13}\sum_{m=1}^{k}\left\|u_m-u_{m-1}\right\|_{H^1(\Omega)}^2\\
\leq& C_{25}+\frac{c_\eta}{4}\max_{1\leq m\leq k}\left\|\frac{\chi_m-\chi_{m-1}}{h_m}\right\|_2^2.
\end{aligned}
\tag{4.27}
$$

By taking the maximum from $m=1$ to $m=K$, we see that (4.15) holds, because of (4.6). □

Lemma 4.4. *There exists a positive constant C_{26} such that*

$$\max_{1\leq m\leq K}\left\|\xi_m\right\|_2+\max_{0\leq m\leq K}\left\|\chi_m\right\|_{H^2(\Omega)}\leq C_{26}. \tag{4.28}$$

Proof. To test (2.4d) by ξ_m, we use the Yosida approximation $\beta^*_{\frac{1}{n}}$ of β^*, which is, see [Bré71, p. 104], a nondecreasing, Lipschitz continuous function on $\mathbb{R}$. The construction of the Yosida approximation and $0\in\beta(0)$ yield that $0=\beta^*_{\frac{1}{n}}(0)$, for all $n\in\mathbb{N}$.
Since $\beta^*_{\frac{1}{n}}$ is the derivative of a convex function on $\mathbb{R}$, we can apply [Bré71, Corollary 13] to show that for every $n\in\mathbb{N}$ there exists a unique $\chi_{m,n}\in H^2(\Omega)$ and a unique $\xi_{m,n}\in L^2(\Omega)$ such that

$$\chi_{m,n}-\varepsilon\Delta\chi_{m,n}+\xi_{m,n}=f_m\quad\text{a.e. in}\quad\Omega, \tag{4.29}$$

$$\xi_{m,n}=\beta^*_{\frac{1}{n}}(\chi_{m,n})\quad\text{a.e. in}\quad\Omega,\qquad\frac{\partial\chi_{m,n}}{\partial n}=0\quad\text{a.e. in}\quad\Gamma, \tag{4.30}$$

with $f_m\in L^2(\Omega)$ defined by

$$f_m:=-\lambda_d'(\chi_m,\chi_{m-1})u_m+\sigma_d'(\chi_m,\chi_{m-1})+\chi_m-\frac{\eta}{h_m}\left(\chi_m-\chi_{m-1}\right). \tag{4.31}$$

Since $\beta^*_{\frac{1}{n}}$ is globally Lipschitz–continuous on $\mathbb{R}$ and, by Sobolev's embedding Theorem, $\chi_{m,n}\in H^{1,6}(\Omega)$, we obtain, by [MM79, Theorem 1], that $\beta^*_{\frac{1}{n}}(\chi_{m,n})=\xi_{m,n}\in H^1(\Omega)$ and, by [MM72, Lemma 2.1 and Remark 2.1], that for this function the generalized chain rules holds. Therefore, since $\beta^*_{\frac{1}{n}}$ is nondecreasing on $\mathbb{R}$, we see that

$$\int_\Omega\nabla(\xi_{m,n})\cdot\nabla\chi_{m,n}\,\mathrm{d}x=\int_\Omega\left(\beta^*_{\frac{1}{n}}\right)'(\chi_{m,n})\left(\nabla\chi_{m,n}\right)^2\,\mathrm{d}x\geq 0. \tag{4.32}$$

We test (4.29) by $\xi_{m,n}$, and use Green's formula, (4.32), (4.30), and Young's inequality, to derive

$$\|\xi_{m,n}\|_2^2 \leq \int_\Omega (f_m - \chi_{m,n})\xi_{m,n}\,\mathrm{d}x \leq \frac{1}{2}\|f_m - \chi_{m,n}\|_2^2 + \frac{1}{2}\|\xi_{m,n}\|_2^2. \tag{4.33}$$

Testing (4.29) by $\chi_{m,n}$ and using Green's formula, (4.30), $0 \in \beta^*_{\frac{1}{n}}(0)$, the monotonicity of $\beta^*_{\frac{1}{n}}$, and Young's inequality, we observe that the sequence $(\chi_{m,n})_{n\in\mathbb{N}}$ is bounded in $H^1(\Omega)$. Hence, the sequence $(\xi_{m,n})_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)$, because of (4.33). Comparing the terms in (4.29), using (4.30) and Lemma AP.4, we see that $(\chi_{m,n})_{n\in\mathbb{N}}$ is also bounded in $H^2(\Omega)$. Thus, there is a $\bar{\chi} \in H^2(\Omega)$ and a $\bar{\xi} \in L^2(\Omega)$ such that, for some subsequences,

$$\chi_{m,n_i} \to \bar{\chi} \quad \text{weakly in} \quad H^2(\Omega), \quad \text{strongly in} \quad H^1(\Omega), \tag{4.34}$$

$$\xi_{m,n_i} \to \bar{\xi} \quad \text{weakly in} \quad L^2(\Omega). \tag{4.35}$$

Now, a passage to the limit in (4.29)–(4.30) and using [Bar76, Cha. II Prob. 1.1(iv)] lead to

$$\bar{\chi} \in D(\beta^*), \quad \bar{\xi} \in \beta^*(\bar{\chi}), \quad \bar{\chi} - \varepsilon\Delta\bar{\chi} + \bar{\xi} = f_m \quad \text{a.e. in} \quad \Omega, \quad \frac{\partial\bar{\chi}}{\partial n} = 0 \quad \text{a.e. in} \quad \Gamma.$$

Since (4.31), (4.3), and (2.4d), yield that (χ_m, ξ_m) is also a solution to this system, which has, by [Bré71, Corollary 13], a unique solution, we see that $\chi_m = \bar{\chi}$ and $\xi_m = \bar{\xi}$. Now, (4.33)–(4.35), (4.31), (4.10), **(A3)**, and (4.15) lead to

$$\frac{1}{2}\|\xi_m\|_2^2 \leq \frac{1}{2}\|f_m - \chi_m\|_2^2 \leq C_{27} + C_{28}\|\lambda_d'(\chi_m, \chi_{m-1})u_m\|_2^2. \tag{4.36}$$

Applying **(A6)**, the generalized Hölder's inequality, $p < 1$, (AP.1), (4.9), and (4.15), we obtain

$$\begin{aligned} &\|\lambda_d'(\chi_m, \chi_{m-1})u_m\|_2 \\ \leq &|\lambda_d'(0,0)|\,\|u_m\|_2 + C_{29}\left(\|\chi_m - 0\|_6 + \|\chi_{m-1} - 0\|_6\right)\left(\|\chi_m^p\|_6 + \left\|\chi_{m-1}^p\right\|_6 + 1\right)\|u_m\|_6 \\ \leq &C_{30}. \end{aligned} \tag{4.37}$$

Comparing the terms in (2.4d), and using **(A3)**, (4.15), (4.10), (4.36), and (4.37), we see that

$$\|\varepsilon\Delta\chi_m\|_2 = \left\|\eta\frac{\chi_m - \chi_{m-1}}{h_m} + \xi_m - \sigma_d'(\chi_m, \chi_{m-1}) + \lambda_d'(\chi_m, \chi_{m-1})\,u_m\right\|_2 \leq C_{31}.$$

Now, using Lemma AP.4, (4.9), and (4.3), we conclude $\|\chi_m\|_{H^2(\Omega)} \leq C_{32}$. Combining this with (4.36), (4.37), and (4.6), we see that (4.28) is satisfied. □

Lemma 4.5. *There exists a positive constant C_{33} such that*

$$\begin{aligned} &\max_{1\leq m\leq K}\left(\|\lambda_d'(\chi_m, \chi_{m-1})\|_\infty + \left\|\frac{\lambda_m - \lambda_{m-1}}{h_m}\right\|_2 + \left\|\frac{\theta_m - \theta_{m-1}}{h_m}\right\|_{H^1(\Omega)^*}\right) \\ &+ \sum_{m=1}^{K} h_m\left\|\frac{\lambda_m - \lambda_{m-1}}{h_m}\right\|_{H^1(\Omega)}^2 + \max_{0\leq m\leq K}\|\lambda_m\|_{H^1(\Omega)} \leq C_{33}. \end{aligned} \tag{4.38}$$

Proof. By looking at the terms in (4.4) and using (4.15) and Lemma 3.2, we see that

$$\max_{1\le m\le K}\left\|c_0\frac{\theta_m-\theta_{m-1}}{h_m}+\frac{\lambda_m-\lambda_{m-1}}{h_m}\right\|_{H^1(\Omega)^*}\le C_{34}. \tag{4.39}$$

Thanks to (4.28), Sobolev's embedding Theorem, and **(A6)**, we have

$$\max_{0\le m\le K}\|\chi_m\|_{H^{1,6}(\Omega)}+\max_{1\le m\le K}\|\lambda_d'(\chi_m,\chi_{m-1})\|_\infty\le C_{35}. \tag{4.40}$$

Combining this with **(A6)**, and [MM79, Theorem 1], we see that $\lambda_d'(\chi_m,\chi_{m-1})\in H^{1,6}(\Omega)$ and

$$\max_{1\le m\le K}\|\nabla\lambda_d'(\chi_m,\chi_{m-1})\|_6\le C_{36}.$$

Therefore, owing to (4.5), Young's inequality, the generalized Hölder's inequality, (4.40), (4.15), and Sobolev's embedding Theorem, we have

$$\begin{aligned}
&\max_{1\le m\le K}\left\|\frac{\lambda_m-\lambda_{m-1}}{h_m}\right\|_2^2+\sum_{m=1}^K h_m\left\|\nabla\left(\frac{\lambda_m-\lambda_{m-1}}{h_m}\right)\right\|_2^2\\
\le&\max_{1\le m\le K}\left(\|\lambda_d'(\chi_m,\chi_{m-1})\|_\infty\left\|\frac{\chi_m-\chi_{m-1}}{h_m}\right\|_2\right)^2\\
&+2\sum_{m=1}^K h_m\|\lambda_d'(\chi_m,\chi_{m-1})\|_\infty^2\left\|\nabla\left(\frac{\chi_m-\chi_{m-1}}{h_m}\right)\right\|_2^2\\
&+2\sum_{m=1}^K h_m\|\nabla\lambda_d'(\chi_m,\chi_{m-1})\|_6^2\left\|\frac{\chi_m-\chi_{m-1}}{h_m}\right\|_3^2\le C_{37}.
\end{aligned}$$

Combining this with (4.39) and (4.6), we see that (4.38) is satisfied. □

Lemma 4.6. *We have $\theta_m\in H^1(\Omega)$ for $0\le m\le K$.*

Proof. We have $\theta_0\in H^1(\Omega)$ by (2.4f) and **(A4)**. For $1\le m\le K$ with $\theta_{m-1}\in H^1(\Omega)$, we define the approximation $\theta_{m,n}\in H^1(\Omega)\cap L^\infty(\Omega)$ for θ_m by

$$\theta_{m,n}:=\left(u_m+\frac{1}{n}\right)^{-1}\quad\text{a.e. in}\quad\Omega,\quad\forall\, n\in\mathbb{N}.$$

The Lebesgue dominated convergence theorem and $\theta_m\in L^2(\Omega)$ yield that

$$\theta_{m,n}\xrightarrow[n\to\infty]{}\theta_m\quad\text{strongly in}\quad L^2(\Omega). \tag{4.41}$$

By applying (4.4) with $v=\theta_{m,n}^3$ and using (4.3), Hölder's inequality, Lemma 3.2, (4.38), (AP.1), and Young's inequality, we see that this sequence is bounded in $H^1(\Omega)$. Combining this with (4.41), we conclude that $\theta_m\in H^1(\Omega)$. □

Lemma 4.7. *There exists a constant C_{38} such that*

$$\max_{0\le m\le K} \|\theta_m\|_2 \le C_{38}. \tag{4.42}$$

Proof. We multiply (2.4c) by h_m and use (4.5). Summing up the resulting equation for $m=1$ to $m=i$, we find

$$c_0\theta_i + \lambda_i + \kappa \sum_{m=1}^{i} h_m \Delta u_m = c_0\theta_0 + \lambda_0 + \sum_{m=1}^{i} h_m g_m \quad \text{a.e. in} \quad \Omega. \tag{4.43}$$

We test (4.43) by $h_i \cdot \Delta u_i$, take the sum from $i=1$ to $i=k$, and apply Green's formula, (2.4e), (4.3), $\theta_m \in H^1(\Omega)$, (AP.3), (AP.2), Lemma 3.2, and Schwarz's inequality, to derive

$$\begin{aligned}
& c_0 \sum_{i=1}^{k} h_i \left\| \frac{\nabla u_i}{u_i} \right\|_2^2 + \frac{\kappa}{2} \left\| \sum_{i=1}^{k} h_i \Delta u_i \right\|_2^2 + \frac{\kappa}{2} \sum_{i=1}^{k} h_i^2 \|\Delta u_i\|_2^2 + \frac{c_0 c_\zeta}{\kappa} \sum_{i=1}^{k} h_i \|\theta_i\|_{L^1(\Gamma)} \\
\le & C_{39} + \int_\Omega \left(\left(c_0\theta_0 + \lambda_0 + \sum_{i=1}^{k} h_i g_i \right) \sum_{i=1}^{k} h_i \Delta u_i \right) \mathrm{d}x - \sum_{i=1}^{k-1} h_{i+1} \int_\Omega g_{i+1} \sum_{m=1}^{i} h_m \Delta u_m \, \mathrm{d}x \\
& + \sum_{i=1}^{k} h_i \int_\Omega \nabla\lambda_i \bullet \nabla u_i \, \mathrm{d}x + \sum_{i=1}^{k} h_i \frac{1}{\kappa} \int_\Gamma \lambda_i \left(\gamma_i u_i - \zeta_i\right) \mathrm{d}\sigma \,.
\end{aligned}$$

Now, by utilizing Young's inequality, (4.6), Lemma 3.2, (4.15), (4.38), and $h_m \le 2h_{m-1}$, we derive

$$\begin{aligned}
& c_0 \sum_{m=1}^{k} h_m \left\| \frac{\nabla u_m}{u_m} \right\|_2^2 + \frac{\kappa}{4} \left\| \sum_{m=1}^{k} h_m \Delta u_m \right\|_2^2 + \frac{\kappa}{2} \sum_{m=1}^{k} h_m^2 \|\Delta u_m\|_2^2 \\
\le & C_{40} + C_{41} \sum_{m=1}^{k-1} h_m \left\| \sum_{i=1}^{m} h_i \Delta u_i \right\|_2^2 .
\end{aligned} \tag{4.44}$$

By applying the discrete version of Gronwall's lemma, we get a uniform upper bound for the left–hand side of (4.44). Looking at the terms in (4.43) and applying (4.38), (4.6), and Lemma 3.2, we see that (4.42) holds. ☐

Lemma 4.8. *There are two positive constant C_{42}, C_{43} such that*

$$\begin{aligned}
& \max_{0\le m\le K} \left(\|u_m\|_{C(\bar\Omega)} + \|\theta_m\|_{C(\bar\Omega)\cap H^1(\Omega)} \right) \\
& + \sum_{m=1}^{K} h_m \left(\left\| \frac{u_m - u_{m-1}}{h_m} \right\|_2^2 + \left\| \frac{\theta_m - \theta_{m-1}}{h_m} \right\|_2^2 \right) \le C_{42},
\end{aligned} \tag{4.45}$$

$$\sum_{m=1}^{K} h_m \|u_m\|_{H^2(\Omega)}^2 \le C_{43}. \tag{4.46}$$

Proof. We deduce, by Lemma 3.2, (4.38), and (AP.1), that

$$\sum_{m=1}^{k} h_m \left\| g_m - \frac{\lambda_m - \lambda_{m-1}}{h_m} \right\|_6^2 \le C_{44}.$$

Thanks to (4.3)–(4.6), (4.15), (4.38), (4.42), and Lemma 3.2, we can apply Moser's technique as in [Kle97a, Lemma 6.11 and 6.12, for $\varepsilon > 0$ fixed], and derive, by using (4.6), that

$$\max_{0 \le m \le K} \left(\|u_m\|_{L^\infty(\Omega)} + \|\theta_m\|_{L^\infty(\Omega)} \right) \le C_{45}.$$

Combining this with $u_m \in H^2(\Omega) \subset C(\bar{\Omega})$, (4.3), (4.15), and Hölder's inequality, we see that (4.45) holds. Now, by looking at the terms in (2.4c), and using (4.5), (4.38), and Lemma 3.2, we see that

$$\sum_{m=1}^{K} h_m \|\Delta u_m\|_2^2 \le C_{46}.$$

Now, Lemma AP.4 yields that (4.46) is satisfied, because of (2.4e), Lemma 3.2, and (4.15). □

Lemma 4.9. *We have*

$$\|\lambda(\chi_k) - \lambda_k\|_{\frac{5}{3}} \le C_{47} |Z|, \quad \forall\, 1 \le k \le K. \tag{4.47}$$

If at least one of the assumptions $\Omega \subset \mathbb{R}^2$ or $\lambda'_d = \lambda'_*$ is satisfied, we have

$$\|\lambda(\chi_k) - \lambda_k\|_2 \le C_{48} |Z|, \quad \forall\, 1 \le k \le K. \tag{4.48}$$

Proof. Applying (4.5), **(A2)**, the mean value theorem, **(A6)**, (4.28), and Sobolev's embedding Theorem, we deduce

$$|\lambda(\chi_k) - \lambda_k| \le C_{49} \sum_{m=1}^{k} h_m^2 \left| \frac{\chi_m - \chi_{m-1}}{h_m} \right|^2 \quad \text{a.e. in} \quad \Omega. \tag{4.49}$$

Hence, recalling Hölder's inequality, the Gagliardo–Nirenberg inequality, and (4.15), we conclude

$$\begin{aligned} \|\lambda(\chi_k) - \lambda_k\|_{\frac{5}{3}}^{\frac{5}{3}} \le & C_{50} |Z|^{\frac{5}{3}} \sum_{m=1}^{k} h_m \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_{\frac{10}{3}}^{\frac{10}{3}} \\ \le & C_{51} |Z|^{\frac{5}{3}} \sum_{m=1}^{k} h_m \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_{H^1(\Omega)}^2 \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_2^{\frac{4}{3}} \le C_{52} |Z|^{\frac{5}{3}}. \end{aligned}$$

Thus, we have shown (4.47).
We use (4.49) and Hölder's inequality to show that

$$\|\lambda(\chi_k) - \lambda_k\|_2^2 \le C_{53} |Z|^2 \sum_{m=1}^{k} h_m \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_4^4.$$

Therefore, if $\Omega \subset \mathbb{R}^2$, recalling the Gagliardo–Nirenberg inequality and (4.15) leads to (4.48). If $\lambda'_d = \lambda'_*$, then (3.2) and (4.5) yield that $\lambda(\chi_k) = \lambda_k$. Hence, (4.48) is satisfied. □

5 Proof of Theorem 2.1 and Corollary 2.1

We assume that **(A1)**–**(A6)** hold.
In the framework of Theorem 2.1, we obtain from **(A6)** that we have positive constants h^* and C_5^* such that (2.7) is satisfied. We assume that $|Z| \leq h^*$.
In the framework of Corollary 2.1, it is part of the assumptions that $|Z| \leq h^*$ where h^* and C_5^* are positive constants fulfilling (2.7).

Because of **(A4)** and Sobolev's embedding Theorem, we see that $\|\chi^0\|_\infty$ is finite.
For any $B > \|\chi^0\|_\infty$, we can consider ϕ^* as in (4.1), β^*, and the corresponding modified version of the time–discrete scheme as in the last section. Lemma 3.3 yields that there exists a solution $\left(\theta_m^B, u_m^B, \chi_m^B, \xi_m^B\right)_{m=0}^K$ to this modified version of the scheme. Since the assumptions used in the last section are satisfied, the estimates derived therein hold for this solution. Now, because of (4.28) and Sobolev's embedding Theorem, there is some positive constant C', independent of B, such that

$$\max_{0\leq m\leq K} \left\|\chi_m^B\right\|_{C(\overline{\Omega})} \leq C'. \tag{5.1}$$

Now, we consider $B := C' + \|\chi^0\|_\infty + 2$. Thanks to (4.1), $\beta^* = \partial\phi^*$, and $\beta = \partial\phi$, we have

$$\beta^*|_{[-C'-1,C'+1]} = \beta|_{[-C'-1,C'+1]}\,.$$

This yields, by (4.3) and (5.1), that the solution to the modified version of scheme is also a solution to the unmodified version of the scheme $(\mathbf{D_Z})$.

It remains to show the uniqueness of the solution. Assume that we have two solutions $\left(\theta_m^{(1)}, u_m^{(1)}, \chi_m^{(1)}, \xi_m^{(1)}\right)_{m=0}^K$ and $\left(\theta_m^{(2)}, u_m^{(2)}, \chi_m^{(2)}, \xi_m^{(2)}\right)_{m=0}^K$ to the scheme $(\mathbf{D_Z})$. Hence, the estimates in the last section are valid for both solutions.
In the sequel, C_i, for $i \in \mathbb{N}$, will always denote positive generic constants, independent of the decomposition Z and the considered solutions.
Thanks to (2.4f), we have $\theta_0^{(1)} = \theta_0^{(2)}$, $u_0^{(1)} = u_0^{(2)}$, $\chi_0^{(1)} = \chi_0^{(2)}$, $\xi_0^{(1)} = \xi_0^{(2)}$ a.e. on Ω.
To prove by induction that the two solutions coincide, we now assume that $1 \leq m \leq K$ is given such that

$$\theta_{m-1}^{(1)} = \theta_{m-1}^{(2)}, \quad u_{m-1}^{(1)} = u_{m-1}^{(2)}, \quad \chi_{m-1}^{(1)} = \chi_{m-1}^{(2)} =: \chi^* \quad \text{a.e. in} \quad \Omega. \tag{5.2}$$

Now, let $u_m := u_m^{(1)} - u_m^{(2)}$ and $\chi_m := \chi_m^{(1)} - \chi_m^{(2)}$.
Using (2.4b), (2.4c), (2.4e), Green's formula, and (5.2), we deduce

$$\begin{aligned} &\theta_m^{(1)} - \theta_m^{(2)} = \frac{-u_m}{u_m^{(1)}u_m^{(2)}} \quad \text{a.e. in} \quad \Omega, \qquad (5.3)\\ &\frac{1}{h_m}\int_\Omega \left(c_0\frac{-u_m}{u_m^{(1)}u_m^{(2)}} + \lambda_d'(\chi_m^{(1)},\chi^*)\left(\chi_m^{(1)}-\chi^*\right) - \lambda_d'(\chi_m^{(2)},\chi^*)\left(\chi_m^{(2)}-\chi^*\right)\right) v\,\mathrm{d}x\\ &-\kappa\int_\Omega \nabla u_m \bullet \nabla v\,\mathrm{d}x - \int_\Gamma \gamma_m u_m v\,\mathrm{d}x \,=0, \quad \forall\, v \in H^1(\Omega). \end{aligned}$$

This yields for $v = -h_m u_m$, by Lemma 3.2,

$$c_0 \left\| \frac{u_m}{\sqrt{u_m^{(1)} u_m^{(2)}}} \right\|_2^2 + h_m C_1 \|u_m\|_{H^1(\Omega)}^2$$

$$\leq \int_\Omega \lambda_d'(\chi_m^{(1)}, \chi^*)\chi_m u_m \,\mathrm{d}x \; + \int_\Omega \left(\lambda_d'(\chi_m^{(1)}, \chi^*) - \lambda_d'(\chi_m^{(2)}, \chi^*)\right)(\chi_m^{(2)} - \chi^*) u_m \,\mathrm{d}x\,. \tag{5.4}$$

Recalling (2.4d) and (5.2), we have

$$\begin{aligned} &\eta \frac{\chi_m}{h_m} - \varepsilon \Delta \chi_m + \xi_m^{(1)} - \xi_m^{(2)} - \sigma_d'(\chi_m^{(1)}, \chi^*) + \sigma_d'(\chi_m^{(2)}, \chi^*) \\ &= -\lambda_d'\left(\chi_m^{(1)}, \chi^*\right) u_m - \left(\lambda_d'\left(\chi_m^{(1)}, \chi^*\right) - \lambda_d'\left(\chi_m^{(2)}, \chi^*\right)\right) u_m^{(2)} \quad \text{a.e. in} \quad \Omega. \end{aligned} \tag{5.5}$$

Testing this equation by χ_m and using **(A3)**, Green's formula, (2.4e), (2.4b), and the monotonicity of β, and adding the resulting estimate to (5.4), we obtain, by (4.45),

$$C_2 \|u_m\|_2^2 + h_m C_1 \|u_m\|_{H^1(\Omega)}^2 + \frac{c_\eta}{h_m}\|\chi_m\|_2^2 + \varepsilon \|\nabla \chi_m\|_2^2 \leq I_1 + I_2, \tag{5.6}$$

$$\text{with} \quad I_1 := \int_\Omega \left(\lambda_d'(\chi_m^{(1)}, \chi^*) - \lambda_d'(\chi_m^{(2)}, \chi^*)\right)\left((\chi_m^{(2)} - \chi^*) u_m - u_m^{(2)} \chi_m\right) \,\mathrm{d}x\,, \tag{5.7}$$

$$I_2 := \int_\Omega \left(\sigma_d'(\chi_m^{(1)}, \chi^*) - \sigma_d'(\chi_m^{(2)}, \chi^*)\right) \chi_m \,\mathrm{d}x\,. \tag{5.8}$$

Now, we consider the framework of Corollary 2.1 and Theorem 2.1 separately.

If we are in the framework of Corollary 2.1, the uniqueness needs only to be shown under the additional assumption that (2.8) holds. Therefore, we have $I_1 = 0$ and

$$I_2 \leq \frac{c_\eta}{2|Z|} \int_\Omega (\chi_m)^2 \,\mathrm{d}x \;\leq \frac{c_\eta}{2h_m}\|\chi_m\|_2^2\,.$$

Hence, (5.6), (5.3), and (5.5) yield that

$$u_m = \chi_m = 0, \quad \theta_m^{(1)} = \theta_m^{(2)}, \quad \xi_m^{(1)} = \xi_m^{(2)} \quad \text{a.e. in} \quad \Omega. \tag{5.9}$$

This finishes the proof of Corollary 2.1.

Now, we consider the framework of Theorem 2.1. **(A6)**, (4.28), and Sobolev's embedding Theorem yield that

$$\left|\lambda_d'(\chi_m^{(1)}, \chi^*) - \lambda_d'(\chi_m^{(2)}, \chi^*)\right| + \left|\sigma_d'(\chi_m^{(1)}, \chi^*) - \sigma_d'(\chi_m^{(2)}, \chi^*)\right| \leq C_3 |\chi_m| \quad \text{a.e. in} \quad \Omega.$$

Hence, by applying the generalized Hölder's inequality, (4.28), (4.45), and Young's inequality, we deduce

$$\begin{aligned} I_1 + I_2 \leq& C_3 \|\chi_m\|_2 \left(\left\|\chi_m^{(2)} - \chi_{m-1}\right\|_\infty \|u_m\|_2 + \left\|u_m^{(2)}\right\|_\infty \|\chi_m\|_2\right) + C_3 \|\chi_m\|_2^2 \\ \leq& \frac{C_2}{2}\|u_m\|_2^2 + C_4 \|\chi_m\|_2^2\,. \end{aligned}$$

Therefore, if we assume that $|Z| \le \frac{c_\eta}{2C_4}$, we obtain $I_1 + I_2 \le \frac{C_2}{2}\|u_m\|_2 + \frac{c_\eta}{2h_m}\|\chi_m\|_2$. Combining this with (5.6), (5.3), and (5.5), we see that (5.9) is satisfied.
Since we have shown that the scheme has a unique solution, if $|Z|$ is sufficiently small, Theorem 2.1 is proved. □

6 Proof of Theorem 2.2 and Theorem 2.3

We assume that **(A1)**–**(A4)** and **(A6)** hold. Thanks to **(A6)**, we have positive constants h^* and C_5^* such that (2.7) is satisfied.

6.1 Properties of the approximations

In this section, we only consider decompositions Z with **(A5)** and $|Z|$ sufficiently small. Hence, Theorem 2.1 yields that there exists a unique solution to the time–discrete scheme $(\mathbf{D_Z})$. Let $\left(\widehat{\theta}^Z, \widehat{u}^Z, \widehat{\chi}^Z, \overline{\xi}^Z\right)$ be the corresponding approximations derived from the solution to $(\mathbf{D_Z})$ as in Remark 2.3.

For $(\lambda_m)_{m=0}^K$ as in (4.5), we define the piecewise linear function $\widehat{\lambda}^Z$ analogously to $\widehat{\chi}^Z$. The piecewise constant functions $\overline{\theta}^Z$, $\overline{u}^Z$, $\overline{\chi}^Z$, $\overline{\gamma}^Z$, $\overline{\zeta}^Z$, $\overline{g}^Z$, $\overline{\lambda}^Z$ are defined analogously to $\overline{\xi}^Z$, and $\underline{\chi}^Z \in L^\infty(0,T;H^2(\Omega))$ is defined by

$$\underline{\chi}^Z(t) = \chi_{m-1}, \quad \forall\, t \in (t_{m-1}, t_m),\ 1 \le m \le K. \tag{6.1}$$

Then, by the definition of the approximations, (2.4a)–(2.4f), and (4.5), we have

$$\widehat{\theta}^Z, \widehat{u}^Z, \widehat{\lambda}^Z \in H^1(0,T;H^1(\Omega)), \quad \overline{u}^Z \in L^2(0,T;H^2(\Omega)),\ \widehat{u}^Z \in L^2(|Z|,T;H^2(\Omega)), \tag{6.2a}$$
$$\widehat{\chi}^Z \in H^1(0,T;H^2(\Omega)), \quad \overline{\chi}^Z, \underline{\chi}^Z \in L^\infty(0,T;H^2(\Omega)), \tag{6.2b}$$
$$\overline{\xi}^Z \in L^\infty(0,T;L^2(\Omega)), \tag{6.2c}$$
$$0 < \widehat{u}^Z,\ 0 < \overline{u}^Z,\ \overline{\theta}^Z = \frac{1}{\overline{u}^Z}, \quad \overline{\chi}^Z, \widehat{\chi}^Z, \underline{\chi}^Z \in D(\beta), \quad \overline{\xi}^Z \in \beta\left(\overline{\chi}^Z\right) \quad \text{a.e. in} \quad \Omega_T, \tag{6.2d}$$
$$c_0\widehat{\theta}^Z_t + \widehat{\lambda}^Z_t + \kappa\Delta\overline{u}^Z = \overline{g}^Z \quad \text{a.e. in} \quad \Omega_T, \tag{6.2e}$$
$$\eta\widehat{\chi}^Z_t - \varepsilon\Delta\overline{\chi}^Z + \overline{\xi}^Z - \sigma_d'(\overline{\chi}^Z, \underline{\chi}^Z) = -\lambda_d'(\overline{\chi}^Z, \underline{\chi}^Z)\overline{u}^Z \quad \text{a.e. in} \quad \Omega_T, \tag{6.2f}$$
$$-\kappa\frac{\partial\overline{u}^Z}{\partial n} = \overline{\gamma}^Z\overline{u}^Z - \overline{\zeta}^Z, \quad \frac{\partial\widehat{\chi}^Z}{\partial n} = 0, \quad \frac{\partial\overline{\chi}^Z}{\partial n} = 0 \quad \text{a.e. in} \quad \Gamma_T, \tag{6.2g}$$
$$\widehat{\theta}^Z(\cdot,0) = \theta^0, \quad \widehat{u}^Z(\cdot,0) = u^0, \quad \widehat{\chi}^Z(\cdot,0) = \chi^0, \quad \widehat{\lambda}^Z(\cdot,0) = \lambda(\chi^0) \quad \text{a.e. in} \quad \Omega. \tag{6.2h}$$

In the sequel, C_i, for $i \in \mathbb{N}$, will always denote positive generic constants, independent of the decomposition Z.

We find, from (4.15), (4.28), (4.38), (4.45), and (4.46):

$$\begin{aligned}
&\left\|\widehat{\theta}^Z\right\|_{W^{1,\infty}(0,T;H^1(\Omega)^*)\cap H^1(0,T;L^2(\Omega))\cap C(\overline{\Omega_T})\cap L^\infty(0,T;H^1(\Omega))} + \left\|\overline{\theta}^Z\right\|_{L^\infty(\Omega_T)\cap L^\infty(0,T;H^1(\Omega))} \\
&+ \left\|\widehat{u}^Z\right\|_{C([0,T];H^1(\Omega))\cap H^1(0,T;L^2(\Omega))\cap C(\overline{\Omega_T})} + \left\|\widehat{u}^Z\right\|_{L^2(|Z|,T;H^2(\Omega))} \\
&+ \left\|\overline{u}^Z\right\|_{L^\infty(0,T;H^1(\Omega))\cap L^\infty(\Omega_T)\cap L^2(0,T;H^2(\Omega))} \le C_1,
\end{aligned} \tag{6.3}$$

$$\left\|\widehat{\chi}^Z\right\|_{W^{1,\infty}(0,T;L^2(\Omega))\cap H^1(0,T;H^1(\Omega))\cap C([0,T];H^2(\Omega))} + \left\|\overline{\chi}^Z\right\|_{L^\infty(0,T;H^2(\Omega))}$$
$$+ \left\|\underline{\chi}^Z\right\|_{L^\infty(0,T;H^2(\Omega))} + \left\|\overline{\xi}^Z\right\|_{L^\infty(0,T;L^2(\Omega))} + \left\|\widehat{\lambda}^Z\right\|_{W^{1,\infty}(0,T;L^2(\Omega))\cap H^1(0,T;H^1(\Omega))} \leq C_2. \quad (6.4)$$

The difference between the piecewise linear and the piecewise constant approximations can be estimated, by using (4.15), **(A2)**, (4.28), Sobolev's embedding Theorem, (4.38), (4.45), and (4.47):

$$\left\|\widehat{\theta}^Z - \overline{\theta}^Z\right\|_{L^2(0,T;L^2(\Omega))\cap L^\infty(0,T;H^1(\Omega)^*)} + \left\|\widehat{u}^Z - \overline{u}^Z\right\|_{L^2(0,T;L^2(\Omega))} \leq C_3\,|Z|\,, \quad (6.5)$$

$$\left\|\widehat{\chi}^Z - \overline{\chi}^Z\right\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} + \left\|\widehat{\chi}^Z - \underline{\chi}^Z\right\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))}$$
$$+ \left\|\lambda(\widehat{\chi}^Z) - \lambda(\overline{\chi}^Z)\right\|_{L^\infty(0,T;L^2(\Omega))} \leq C_4\,|Z|\,, \quad (6.6)$$

$$\left\|\widehat{\lambda}^Z - \overline{\lambda}^Z\right\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} + \left\|\lambda(\overline{\chi}^Z) - \overline{\lambda}^Z\right\|_{L^\infty(0,T;L^{\frac{5}{3}}\Omega)} \leq C_5\,|Z|\,, \quad (6.7)$$

$$\left\|\widehat{u}^Z - \overline{u}^Z\right\|_{L^2(0,T;H^1(\Omega))} \leq C_6\sqrt{|Z|}. \quad (6.8)$$

For the approximation of the data, we have, by **(A3)**:

Lemma 6.1. *The functions* $\overline{g}^Z, \overline{\gamma}^Z, \overline{\zeta}^Z$ *fulfill*

$$\left\|\overline{g}^Z\right\|_{L^\infty(\Omega_T)} + \left\|\overline{\gamma}^Z\right\|_{L^\infty(0,T;C^1(\Gamma))} + \left\|\overline{\zeta}^Z\right\|_{L^\infty(\Gamma_T)\cap L^\infty(0,T;H^{\frac{1}{2}}(\Gamma))} \leq C_7, \quad (6.9)$$

$$\left\|g - \overline{g}^Z\right\|_{L^2(0,T;L^\infty(\Omega))} + \left\|\gamma - \overline{\gamma}^Z\right\|_{L^\infty(\Gamma_T)} + \left\|\zeta - \overline{\zeta}^Z\right\|_{L^2(0,T;L^2(\Gamma))} \leq C_8\,|Z|\,. \quad (6.10)$$

Now, estimates similar to [NSV] are used to prove the following lemma, which is important to improve the order of the error estimate from $\sqrt{|Z|}$ to $|Z|$.

Lemma 6.2. *We have a positive constant* C_9 *such that*

$$-\int_0^s\int_\Omega \left(\xi - \overline{\xi}^Z\right)\left(\chi - \widehat{\chi}^Z\right)\,\mathrm{d}x\;\mathrm{d}t \leq C_9\,|Z|^2\,, \quad \forall\, s\in[0,T], \quad (6.11)$$

for all $\chi, \xi \in L^2(0,T;L^2(\Omega))$ *with*

$$\chi \in D(\beta), \quad \xi \in \beta(\chi) \quad \textit{a.e. in} \quad \Omega_T. \quad (6.12)$$

Proof. From (6.12), (6.2d), and $\beta = \partial\phi$, we get

$$A_1 := -\int_0^s\int_\Omega \left(\xi - \overline{\xi}^Z\right)\left(\chi - \widehat{\chi}^Z\right)\,\mathrm{d}x\;\mathrm{d}t \leq \int_0^s\int_\Omega \left(-\phi(\overline{\chi}^Z) + \overline{\xi}^Z\left(\overline{\chi}^Z - \widehat{\chi}^Z\right) + \phi(\widehat{\chi}^Z)\right)\mathrm{d}x\;\mathrm{d}t\,.$$

For $l^Z : (0,T] \to [0,1]$ defined by

$$l^Z(t) = \frac{t_m - t}{h_m}, \qquad \forall\, t\in(t_{m-1},t_m],\ 1\leq m\leq K, \quad (6.13)$$

holds $\quad \widehat{\chi}^Z = \left(1 - l^Z\right)\overline{\chi}^Z + l^Z\underline{\chi}^Z = \overline{\chi}^Z + l^Z\left(\underline{\chi}^Z - \overline{\chi}^Z\right) \quad$ a.e. in $\quad \Omega_T$.

We apply the convexity of ϕ, to show that

$$A_1 \leq \int_0^s l^Z \int_\Omega \left(-\phi(\overline{\chi}^Z) + \phi(\underline{\chi}^Z) + \overline{\xi}^Z \left(\overline{\chi}^Z - \underline{\chi}^Z\right)\right) \, \mathrm{d}x \; \mathrm{d}t \, .$$

Since (6.2d) and $\beta = \partial\phi$ yield that the integrand is a.e. non–negative, we see, by (6.13), (2.4b), (2.4f), **(A4)**, and $\beta = \partial\phi$, that

$$\begin{aligned} A_1 &\leq \sum_{m=1}^K \int_{t_{m-1}}^{t_m} \frac{t_m - t}{h_m} \, \mathrm{d}t \int_\Omega \left(-\phi(\chi_m) + \phi(\chi_{m-1}) + \xi_m \left(\chi_m - \chi_{m-1}\right)\right) \, \mathrm{d}x \\ &\leq \frac{1}{2} |Z|^2 \sum_{m=1}^K \left\| \left(\xi_m - \xi_{m-1}\right) \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_1 . \end{aligned}$$

Hence, (6.11) holds because of (4.15). □

6.2 Error estimates

Now, we estimate the difference between the approximation and one exact solution. Here, ideas from [CS97, Col96, Kle97a, NSV] are used.

Lemma 6.3. *For every solution (θ, u, χ, ξ) to the Penrose–Fife system* ***(PF)*** *there are positive constants C_{10}, C_{11} such that*

$$\begin{aligned} &\max_{0 \leq s \leq T} \left\| \int_0^s \left(u(\tau) - \overline{u}^Z(\tau)\right) \, \mathrm{d}\tau \right\|^2_{H^1(\Omega)} + \max_{0 \leq s \leq T} \left\| \int_0^s \gamma(\tau) \left(u(\tau) - \overline{u}^Z(\tau)\right) \, \mathrm{d}\tau \right\|^2_\Gamma \\ &+ \left\| \frac{u - \overline{u}^Z}{\sqrt{u \overline{u}^Z}} \right\|^2_{L^2(0,T;L^2(\Omega))} + \left\| u - \overline{u}^Z \right\|^2_{L^2(0,T;L^{\frac{3}{2}}(\Omega))} + \left\| \theta - \overline{\theta}^Z \right\|^2_{L^2(0,T;L^1(\Omega))} \\ &+ \left\| \nabla \left(\chi - \overline{\chi}^Z\right) \right\|^2_{L^2(0,T;L^2(\Omega))} + \left\| \chi - \widehat{\chi}^Z \right\|^2_{L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))} \\ &\leq C_{10} A^Z + C_{11} \left(|Z|^2 + |Z| \left\| u - \overline{u}^Z \right\|_{L^2(0,T;L^2(\Omega))}\right) \qquad (6.14) \\ &\longrightarrow 0, \quad \textit{as} \quad |Z| \longrightarrow 0, \qquad (6.15) \end{aligned}$$

$$\begin{aligned} \textit{with} \quad A^Z &:= \int_0^T \left\| \left(\lambda(\widehat{\chi}^Z) - \widehat{\lambda}^Z\right) \left(u - \overline{u}^Z\right) \right\|_1 \, \mathrm{d}t \qquad (6.16) \\ &\leq C_{12} |Z| \left\| u - \overline{u}^Z \right\|^{\frac{17}{20}}_{L^2(0,T;L^2(\Omega))} . \qquad (6.17) \end{aligned}$$

Proof. The generic constants may depend on the solution to the Penrose–Fife system. Thanks to (2.1a), (2.1b), Sobolev's embedding Theorem, and **(A2)**, we have

$$\begin{aligned} &\|\theta\|_{L^\infty(0,T;L^2(\Omega))} + \|u\|_{L^\infty(0,T;H^1(\Omega))} + \|u\|_{L^2(0,T;H^2(\Omega))} \\ &+ \|\chi\|_{L^\infty(\Omega_T)} + \|\lambda'(\chi)\|_{L^\infty(\Omega_T)} \leq C_{13}. \qquad (6.18) \end{aligned}$$

First, we work on the equation for θ and u. Integrating the difference of (2.1e) and (6.2e) in time, and testing the corresponding equation by v, and using (2.1g), (2.1h), (6.2g), and (6.2h), we obtain for all $v \in H^1(\Omega)$,

$$\begin{aligned}
&\int_\Omega \Big(c_0(\theta(t) - \widehat{\theta}^Z(t)) + \lambda(\chi(t)) - \widehat{\lambda}^Z(t)\Big)\, v \,\mathrm{d}x \;-\kappa \int_0^t \int_\Omega \nabla\left(u(\tau) - \overline{u}^Z(\tau)\right) \bullet \nabla v \,\mathrm{d}x \;\mathrm{d}\tau \\
&= \int_\Omega \int_0^t \left(g(\tau) - \overline{g}^Z(\tau)\right)\, \mathrm{d}\tau \, v \,\mathrm{d}x \;+ \int_0^t \int_\Gamma \gamma(\tau)\left(u(\tau) - \overline{u}^Z(\tau)\right) v \,\mathrm{d}\sigma \;\mathrm{d}\tau \\
&+ \int_0^t \int_\Gamma \Big(\left(\gamma(\tau) - \overline{\gamma}^Z(\tau)\right) \overline{u}^Z(\tau) - \Big(\zeta(\tau) - \overline{\zeta}^Z(\tau)\Big)\Big)\, v \,\mathrm{d}\sigma \;\mathrm{d}\tau\,, \quad \forall\, t \in (0,T).
\end{aligned} \tag{6.19}$$

For a.e. $t \in (0,T)$, this yields, with $v = -\left(u(t) - \overline{u}^Z(t)\right)$, by (2.1d) and (6.2d),

$$\begin{aligned}
&\int_\Omega c_0 \left(\frac{\left(u - \overline{u}^Z\right)^2}{u\overline{u}^Z} - \left(\overline{\theta}^Z - \widehat{\theta}^Z\right)\left(u - \overline{u}^Z\right)\right)\,\mathrm{d}x \;- \int_\Omega \left(\lambda(\chi) - \widehat{\lambda}^Z\right)\left(u - \overline{u}^Z\right)\,\mathrm{d}x \\
&= - \int_\Omega \int_0^t \left(g(\tau) - \overline{g}^Z(\tau)\right)\,\mathrm{d}\tau\;\left(u - \overline{u}^Z\right)\,\mathrm{d}x \\
&\quad - \int_\Gamma \left(\int_0^t \gamma(\tau)\left(u(\tau) - \overline{u}^Z(\tau)\right)\,\mathrm{d}\tau\right)\left(u - \overline{u}^Z\right)\,\mathrm{d}\sigma \\
&\quad - \int_\Gamma \int_0^t \Big(\left(\gamma(\tau) - \overline{\gamma}^Z(\tau)\right)\overline{u}^Z(\tau) \;- \Big(\zeta(\tau) - \overline{\zeta}^Z(\tau)\Big)\Big)\,\mathrm{d}\tau\;\left(u - \overline{u}^Z\right)\,\mathrm{d}\sigma \\
&\quad - \kappa \int_\Omega \int_0^t \nabla\left(u(\tau) - \overline{u}^Z(\tau)\right)\,\mathrm{d}\tau \;\bullet \nabla\left(u - \overline{u}^Z\right)\,\mathrm{d}x \;=: A_2 + A_3 + A_4 + A_5.
\end{aligned} \tag{6.20}$$

Owing to (6.2d), (2.1d), the generalized Hölder's inequality, (AP.1), (6.3), and (6.18), we see that

$$\int_0^s \left(\left\|u - \overline{u}^Z\right\|_{\frac{3}{2}}^2 + \left\|\theta - \overline{\theta}^Z\right\|_1^2\right)\,\mathrm{d}t \;\leq C_{14} \int_0^s \left\|\frac{u - \overline{u}^Z}{\sqrt{u\overline{u}^Z}}\right\|_2^2 \,\mathrm{d}t\,. \tag{6.21}$$

We have, by Hölder's inequality, (6.10), and Young's inequality,

$$A_2 \leq C_{15} \left\|g - \overline{g}^Z\right\|_{L^2(0,T;L^\infty(\Omega))} \left\|u - \overline{u}^Z\right\|_{\frac{3}{2}} \leq C_{16}\, |Z|^2 + \frac{c_0}{4C_{14}} \left\|u - \overline{u}^Z\right\|_{\frac{3}{2}}^2, \tag{6.22}$$

$$
\begin{aligned}
A_3 = & -\frac{1}{2}\partial_t \left\| \frac{1}{\sqrt{\gamma(t)}} \int_0^t \gamma(\tau)\left(u(\tau) - \overline{u}^Z(\tau)\right)\, \mathrm{d}\tau \right\|^2_{L^2(\Gamma)} \\
& - \int_\Gamma \frac{\gamma_t(t)}{2(\gamma(t))^2} \left(\int_0^t \gamma(\tau)\left(u(\tau) - \overline{u}^Z(\tau)\right)\, \mathrm{d}\tau \right)^2 \mathrm{d}\sigma\,, \qquad (6.23) \\
A_5 = & -\frac{\kappa}{2}\partial_t \left\| \int_0^t \nabla\left(u(\tau) - \overline{u}^Z(\tau)\right)\, \mathrm{d}\tau \right\|^2_2 . \qquad (6.24)
\end{aligned}
$$

By integrating (6.20) from 0 to s and using (6.16), (6.21)–(6.23), we obtain

$$
\begin{aligned}
& \frac{c_0}{2C_{14}} \int_0^s \left(\frac{1}{2}\left\|u - \overline{u}^Z\right\|^2_{\frac{3}{2}} + \left\|\theta - \overline{\theta}^Z\right\|^2_1 \right) \mathrm{d}t \,+ \frac{c_0}{2} \int_0^s \left\| \frac{u - \overline{u}^Z}{\sqrt{u\overline{u}^Z}} \right\|^2_2 \mathrm{d}t \\
& + \frac{\kappa}{2} \left\| \nabla \int_0^s \left(u(\tau) - \overline{u}^Z(\tau)\right)\, \mathrm{d}\tau \right\|^2_2 + \frac{1}{2} \left\| \frac{1}{\sqrt{\gamma(s)}} \int_0^s \gamma(\tau)\left(u(\tau) - \overline{u}^Z(\tau)\right)\, \mathrm{d}\tau \right\|^2_{L^2(\Gamma)} \\
\le & \int_0^s \int_\Omega \left(\lambda(\chi) - \lambda(\widehat{\chi}^Z)\right)\left(u - \overline{u}^Z\right)\, \mathrm{d}x\ \mathrm{d}t \,+ A^Z + c_0 \int_0^s \int_\Omega \left(\overline{\theta}^Z - \widehat{\theta}^Z\right)\left(u - \overline{u}^Z\right)\, \mathrm{d}x\ \mathrm{d}t \\
& + \int_0^s \int_\Gamma \int_0^t \left(\zeta(\tau) - \overline{\zeta}^Z(\tau) - \left(\gamma(\tau) - \overline{\gamma}^Z(\tau)\right)\overline{u}^Z(\tau)\right) \mathrm{d}\tau\ \left(u - \overline{u}^Z\right)\, \mathrm{d}\sigma\ \mathrm{d}t \\
& + TC_{16}\left|Z\right|^2 - \int_0^s \int_\Gamma \frac{\gamma_t(t)}{2(\gamma(t))^2} \left(\int_0^t \gamma(\tau)\left(u(\tau) - \overline{u}^Z(\tau)\right)\, \mathrm{d}\tau \right)^2 \mathrm{d}\sigma\ \mathrm{d}t \\
=: & \; A_6 + A^Z + A_7 + A_8 + TC_{16}\left|Z\right|^2 + A_9. \qquad (6.25)
\end{aligned}
$$

Applying Poincaré's inequality and Hölder's inequality, we conclude that

$$
\begin{aligned}
& C_{17} \left\| \int_0^s \left(u(\tau) - \overline{u}^Z(\tau)\right)\, \mathrm{d}\tau \right\|^2_{H^1(\Omega)} \\
\le & \frac{\kappa}{2} \left\| \nabla \int_0^s \left(u(\tau) - \overline{u}^Z(\tau)\right)\, \mathrm{d}\tau \right\|^2_2 + \frac{c_0}{8C_{14}} \int_0^s \left\|u(\tau) - \overline{u}^Z(\tau)\right\|^2_{\frac{3}{2}}\, \mathrm{d}\tau\,. \qquad (6.26)
\end{aligned}
$$

Using Hölder's inequality, **(A2)**, (6.18), (6.4), Sobolev's embedding Theorem, and (6.5), we derive

$$
A_6 + A_7 \le C_{18} \int_0^s \left\| \left(\chi - \widehat{\chi}^Z\right)\left(u - \overline{u}^Z\right) \right\|_1 \mathrm{d}t \,+ C_{19}\left|Z\right| \left\|u - \overline{u}^Z\right\|_{L^2(0,T;L^2(\Omega))} . \qquad (6.27)
$$

Partial integration with respect to time and Hölder's inequality results in

$$A_8 \leq \left(\left\| \int_0^s \left(\zeta - \overline{\zeta}^Z \right) \, \mathrm{d}t \right\|_{L^2(\Gamma)} + \int_0^s \left\| \gamma - \overline{\gamma}^Z \right\|_{L^\infty(\Gamma)} \, \mathrm{d}t \, \left\| \overline{u}^Z \right\|_{L^\infty(0,T;L^2(\Gamma))} \right)$$
$$\left\| \int_0^s \left(u(\tau) - \overline{u}^Z(\tau) \right) \, \mathrm{d}\tau \right\|_{L^2(\Gamma)}$$
$$+ \int_0^s \left(\left\| \zeta - \overline{\zeta}^Z \right\|_{L^2(\Gamma)} + \left\| \gamma - \overline{\gamma}^Z \right\|_{L^\infty(\Gamma)} \left\| \overline{u}^Z \right\|_{L^2(\Gamma)} \right) \left\| \int_0^t \left(u(\tau) - \overline{u}^Z(\tau) \right) \, \mathrm{d}\tau \right\|_{L^2(\Gamma)} \mathrm{d}t \, .$$

Because of the trace theorem, (6.3), (6.10), and Young's inequality, we observe

$$A_8 \leq \frac{C_{17}}{2} \left\| \int_0^s \left(u(\tau) - \overline{u}^Z(\tau) \right) \, \mathrm{d}\tau \right\|_{H^1(\Omega)}^2 + \frac{1}{2} \int_0^s \left\| \int_0^t \left(u(\tau) - \overline{u}^Z(\tau) \right) \, \mathrm{d}\tau \right\|_{H^1(\Omega)}^2 \mathrm{d}t + C_{20} \left| Z \right|^2 . \tag{6.28}$$

In the light of Hölder's inequality and **(A3)**, we see

$$A_9 \leq C_{21} \int_0^s \left\| \frac{1}{\sqrt{\gamma(t)}} \int_0^t \gamma(\tau) \left(u(\tau) - \overline{u}^Z(\tau) \right) \, \mathrm{d}\tau \right\|_{L^2(\Gamma)}^2 \mathrm{d}t \, . \tag{6.29}$$

Hence, we get, by using Hölder's inequality, (6.25)–(6.29), and Young's inequality,

$$\frac{C_{17}}{2} \left\| \int_0^s \left(u(\tau) - \overline{u}^Z(\tau) \right) \, \mathrm{d}\tau \right\|_{H^1(\Omega)}^2 + \frac{c_0}{2C_{14}} \int_0^s \left(\frac{1}{4} \left\| u - \overline{u}^Z \right\|_{\frac{3}{2}}^2 + \left\| \theta - \overline{\theta}^Z \right\|_1^2 \right) \mathrm{d}t$$
$$+ \frac{c_0}{2} \int_0^s \left\| \frac{u - \overline{u}^Z}{\sqrt{u \overline{u}^Z}} \right\|_2^2 \mathrm{d}t + \frac{1}{2} \left\| \frac{1}{\sqrt{\gamma(s)}} \int_0^s \gamma(\tau) \left(u(\tau) - \overline{u}^Z(\tau) \right) \, \mathrm{d}\tau \right\|_{L^2(\Gamma)}^2$$
$$\leq C_{18} \int_0^s \left\| \left(\chi - \widehat{\chi}^Z \right) \left(u - \overline{u}^Z \right) \right\|_1 \mathrm{d}t + A^Z + C_{19} \left| Z \right| \left\| u - \overline{u}^Z \right\|_{L^2(0,T;L^2(\Omega))}$$
$$+ \frac{1}{2} \int_0^s \left\| \int_0^t \left(u(\tau) - \overline{u}^Z(\tau) \right) \, \mathrm{d}\tau \right\|_{H^1(\Omega)}^2 \mathrm{d}t$$
$$+ \left(C_{20} + T C_{16} \right) \left| Z \right|^2 + C_{21} \int_0^s \left\| \frac{1}{\sqrt{\gamma(t)}} \int_0^t \gamma(\tau) \left(u(\tau) - \overline{u}^Z(\tau) \right) \, \mathrm{d}\tau \right\|_{L^2(\Gamma)}^2 \mathrm{d}t \, . \tag{6.30}$$

Now, estimates for χ will be derived. Subtracting (6.2f) from (2.1f), we obtain that

$$\begin{aligned}&\eta(\chi_t-\widehat{\chi}_t^Z)-\varepsilon\Delta\left(\chi-\overline{\chi}^Z\right)+\xi-\overline{\xi}^Z-\sigma'(\chi)+\sigma_d'(\overline{\chi}^Z,\underline{\chi}^Z)\\&=-\lambda'(\chi)u+\lambda_d'(\overline{\chi}^Z,\underline{\chi}^Z)\overline{u}^Z\quad\text{a.e. in}\quad\Omega_T.\end{aligned}\tag{6.31}$$

Testing this with $\chi-\widehat{\chi}^Z$ and recalling **(A3)**, (2.1g), and (6.2g), we end up with

$$\begin{aligned}&\frac{1}{2}\partial_t\left\|\sqrt{\eta}\,\left(\chi-\widehat{\chi}^Z\right)\right\|_2^2+\varepsilon\int_\Omega\nabla\left(\chi-\overline{\chi}^Z\right)\bullet\nabla\left(\chi-\widehat{\chi}^Z\right)\,\mathrm{d}x+\int_\Omega\left(\xi-\overline{\xi}^Z\right)\left(\chi-\widehat{\chi}^Z\right)\,\mathrm{d}x\\&\leq\int_\Omega\left(\sigma'(\chi)-\sigma_d'(\overline{\chi}^Z,\underline{\chi}^Z)\right)\left(\chi-\widehat{\chi}^Z\right)\,\mathrm{d}x-\int_\Omega\left(\lambda'(\chi)u-\lambda_d'(\overline{\chi}^Z,\underline{\chi}^Z)\overline{u}^Z\right)\left(\chi-\widehat{\chi}^Z\right)\,\mathrm{d}x\\&=:\ A_{10}+A_{11}.\end{aligned}\tag{6.32}$$

We have

$$\begin{aligned}&\varepsilon\int_\Omega\nabla\left(\chi-\overline{\chi}^Z\right)\bullet\nabla\left(\chi-\widehat{\chi}^Z\right)\,\mathrm{d}x\\&=\frac{\varepsilon}{2}\left\|\nabla\left(\chi-\overline{\chi}^Z\right)\right\|_2^2+\frac{\varepsilon}{2}\left\|\nabla\left(\chi-\widehat{\chi}^Z\right)\right\|_2^2-\frac{\varepsilon}{2}\left\|\nabla\left(\overline{\chi}^Z-\widehat{\chi}^Z\right)\right\|_2^2.\end{aligned}\tag{6.33}$$

Using (6.32), **(A6)**, **(A2)**, (6.18), (6.4), Sobolev's embedding Theorem, Hölder's inequality, (6.6), and Young's inequality, we conclude

$$\begin{aligned}A_{10}&=\int_\Omega\left(\sigma'(\chi)-\sigma'(\widehat{\chi}^Z)\right)\left(\chi-\widehat{\chi}^Z\right)\,\mathrm{d}x+\int_\Omega\left(\sigma_d'(\widehat{\chi}^Z,\widehat{\chi}^Z)-\sigma_d'(\overline{\chi}^Z,\underline{\chi}^Z)\right)\left(\chi-\widehat{\chi}^Z\right)\,\mathrm{d}x\\&\leq C_{22}\left\|\chi-\widehat{\chi}^Z\right\|_2^2+C_{23}\left|Z\right|^2\quad\text{a.e. in}\quad(0,T).\end{aligned}\tag{6.34}$$

In the light of (6.32), **(A6)**, the generalized Hölder's inequality, **(A2)**, (6.18), (6.3), (6.4), Sobolev's embedding Theorem, (6.6), and Young's inequality, we see that

$$\begin{aligned}A_{11}=&-\int_\Omega\Big(\lambda'(\chi)\left(u-\overline{u}^Z\right)+\left(\lambda'(\chi)-\lambda'(\widehat{\chi}^Z)\right)\overline{u}^Z\Big)\left(\chi-\widehat{\chi}^Z\right)\,\mathrm{d}x\\&-\int_\Omega\Big(\lambda_d'(\widehat{\chi}^Z,\widehat{\chi}^Z)-\lambda_d'(\overline{\chi}^Z,\underline{\chi}^Z)\Big)\overline{u}^Z\left(\chi-\widehat{\chi}^Z\right)\,\mathrm{d}x\\\leq&C_{24}\left\|\left(u-\overline{u}^Z\right)\left(\chi-\widehat{\chi}^Z\right)\right\|_1+C_{25}\left\|\chi-\widehat{\chi}^Z\right\|_2^2+C_{26}\left|Z\right|^2.\end{aligned}\tag{6.35}$$

Combining (6.32)–(6.35), integrating in time, using **(A3)**, (2.1h), (6.2h), (6.11), (6.6), and adding the resulting estimate to (6.30), we get

$$\begin{aligned}
&\frac{C_{17}}{2}\left\|\int_0^s \left(u(\tau)-\overline{u}^Z(\tau)\right)\,\mathrm{d}\tau\right\|^2_{H^1(\Omega)} + C_{27}\int_0^s \left(\left\|u-\overline{u}^Z\right\|^2_{\frac{3}{2}} + \left\|\theta-\overline{\theta}^Z\right\|^2_{L^1(\Omega)}\right)\,\mathrm{d}t \\
&+\frac{c_0}{2}\int_0^s \left\|\frac{u-\overline{u}^Z}{\sqrt{u\overline{u}^Z}}\right\|_2^2\,\mathrm{d}t + \frac{1}{2}\left\|\frac{1}{\sqrt{\gamma(s)}}\int_0^s \gamma(\tau)\left(u(\tau)-\overline{u}^Z(\tau)\right)\,\mathrm{d}\tau\right\|^2_{L^2(\Gamma)} \\
&+\frac{c_\eta}{2}\left\|\chi(s)-\widehat{\chi}^Z(s)\right\|_2^2 + \frac{\varepsilon}{2}\int_0^s \left\|\nabla\left(\chi-\overline{\chi}^Z\right)\right\|_2^2\,\mathrm{d}t + \frac{\varepsilon}{2}\int_0^s \left\|\chi-\widehat{\chi}^Z\right\|^2_{H^1(\Omega)}\,\mathrm{d}t \\
\leq& A_{12} + A^Z + C_{19}\left|Z\right|\left\|u-\overline{u}^Z\right\|_{L^2(0,T;L^2(\Omega))} + C_{28}\left|Z\right|^2 \\
&+\frac{1}{2}\int_0^s \left\|\int_0^t \left(u(\tau)-\overline{u}^Z(\tau)\right)\,\mathrm{d}\tau\right\|^2_{H^1(\Omega)}\,\mathrm{d}t + \left(\frac{\varepsilon}{2}+C_{22}+C_{25}\right)\int_0^s \left\|\chi-\widehat{\chi}^Z\right\|_2^2\,\mathrm{d}t \\
&+C_{21}\int_0^s \left\|\frac{1}{\sqrt{\gamma(t)}}\int_0^t \gamma(\tau)\left(u(\tau)-\overline{u}^Z(\tau)\right)\,\mathrm{d}\tau\right\|^2_{L^2(\Gamma)}\,\mathrm{d}t
\end{aligned}\tag{6.36}$$

with

$$A_{12} := \left(C_{18}+C_{24}\right)\int_0^s \left\|\left(\chi-\widehat{\chi}^Z\right)\left(u-\overline{u}^Z\right)\right\|_1\,\mathrm{d}t\,.$$

Using Hölder's inequality, Young's inequality, and the Gagliardo–Nirenberg inequality, we obtain

$$A_{12} \leq \frac{C_{27}}{2}\int_0^s \left\|u-\overline{u}^Z\right\|^2_{\frac{3}{2}}\,\mathrm{d}t + \frac{\varepsilon}{4}\int_0^s \left\|\chi-\widehat{\chi}^Z\right\|^2_{H^1(\Omega)}\,\mathrm{d}t + C_{29}\int_0^s \left\|\chi-\widehat{\chi}^Z\right\|_2^2\,\mathrm{d}t\,.$$

Hence, (6.36), Gronwall's lemma, and **(A3)** yield that (6.14) is satisfied.
By applying (6.16), Hölder's inequality, (6.6), (6.7), and the Gagliardo–Nirenberg inequality, we get

$$A^Z \leq \int_0^T \left\|\lambda(\widehat{\chi}^Z)-\widehat{\lambda}^Z\right\|_{\frac{5}{3}}\left\|u-\overline{u}^Z\right\|_{\frac{5}{2}}\,\mathrm{d}t \leq C_{30}\left|Z\right|\int_0^T \left\|u-\overline{u}^Z\right\|^{\frac{3}{20}}_{H^2(\Omega)}\left\|u-\overline{u}^Z\right\|_2^{\frac{17}{20}}\,\mathrm{d}t\,.$$

Hence, using Hölder's inequality, (6.18), and (6.3), we deduce that (6.17) and (6.15) are satisfied. □

6.3 Proof of Theorem 2.2

Proof. Thanks to the estimates (6.3), (6.4), Sobolev's embedding Theorem, and compactness (see, e.g., [Zei90, Prop. 23.7, 23.19, Prob. 23.12]), we get $(\theta, u, \chi, \xi, \lambda^*)$ fulfilling (2.1b)–(2.1c), (2.9)–(2.11), and

$$\theta \in H^1(0,T;L^2(\Omega)), \quad u \in L^\infty(0,T;H^1(\Omega)), \quad \lambda^* \in W^{1,\infty}(0,T;L^2(\Omega)).$$

such that we have, for some subsequence with $|Z| \to 0$, the convergences (2.12)–(2.20), and

$$\widehat{\lambda}^Z \longrightarrow \lambda^* \quad \text{weakly–star in} \quad W^{1,\infty}(0,T;L^2(\Omega)). \tag{6.37}$$

We obtain the convergences (2.12)–(2.20) for the whole sequence, if we can show that (θ, u, χ, ξ) is the unique solution to the Penrose–Fife system **(PF)**. Hence, we need only to prove this, to finish the proof of Theorem 2.2.

Thanks to the convergences for $\widehat{\chi}^Z$ in (2.18), (6.4), the Aubin compactness lemma (see, e.g., [Lio69, p. 58]), and (6.6), we also get

$$\widehat{\chi}^Z \longrightarrow \chi, \quad \overline{\chi}^Z \longrightarrow \chi, \quad \underline{\chi}^Z \longrightarrow \chi \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)). \tag{6.38}$$

Hence, after possibly extracting a further subsequence, we have

$$\overline{\chi}^Z \longrightarrow \chi, \quad \underline{\chi}^Z \longrightarrow \chi \qquad \text{a.e. in} \quad \Omega_T.$$

This yields, thanks to **(A2)**, **(A6)**, (6.4), and the Lebesgue dominated convergence theorem, that

$$\lambda(\overline{\chi}^Z) \longrightarrow \lambda(\chi), \quad \lambda_d'(\overline{\chi}^Z, \underline{\chi}^Z) \longrightarrow \lambda'(\chi), \quad \sigma_d'(\overline{\chi}^Z, \underline{\chi}^Z) \longrightarrow \sigma'(\chi) \quad \text{strongly in} \quad L^2(\Omega_T). \tag{6.39}$$

Thus, (6.37), (6.6), and (6.7) yield that $\lambda^* = \lambda(\chi)$ a.e. on Ω_T. Hence, using (2.12)–(2.20), (6.37)–(6.39), and (6.3)–(6.10), we can pass to the limit in (6.2a)–(6.2h) and obtain that (θ, u, χ, ξ) is a solution to the Penrose–Fife system **(PF)**. Details can be found in [Kle97a, Sec. 8]. It remains to show that this solution is unique.
Let $(\theta^*, u^*, \chi^*, \xi^*)$ be any solution to the Penrose–Fife system **(PF)**. Since we can apply Lemma 6.3 for this solution, using (6.15) and the convergences (2.12)–(2.19) yields that

$$\theta^* = \theta, \quad u^* = u, \quad \chi^* = \chi \quad \text{a.e. in} \quad \Omega_T.$$

Comparing the terms in (2.1f), we see that the two solutions coincide. □

6.4 Proof of Theorem 2.3

Proof. Thanks to (2.1d), (6.2d), Hölder's inequality, (2.9), (2.10), and (6.3), we have

$$\left\|u - \overline{u}^Z\right\|^2_{L^2(0,T;L^2(\Omega))} + \left\|\theta - \overline{\theta}^Z\right\|^2_{L^2(0,T;L^2(\Omega))} \le C_{31} \left\|\frac{u - \overline{u}^Z}{\sqrt{u\overline{u}^Z}}\right\|_{L^2(0,T;L^2(\Omega))}.$$

Moreover, we have $\chi - \widehat{\chi}^Z \in C([0,T];L^2(\Omega))$, because of (6.2b) and (2.1b). Hence, we obtain from (6.14) and Young's inequality that

$$\begin{aligned} A_{13} := \max_{0\le s\le T} \left\| \int_0^s \left(u(\tau) - \overline{u}^Z(\tau)\right) \,\mathrm{d}\tau \right\|^2_{H^1(\Omega)} + \max_{0\le s\le T} \left\| \int_0^s \gamma(\tau) \left(u(\tau) - \overline{u}^Z(\tau)\right) \,\mathrm{d}\tau \right\|^2_{\Gamma} \\ + \frac{1}{2C_{31}} \left\|u - \overline{u}^Z\right\|^2_{L^2(0,T;L^2(\Omega))} + \frac{1}{C_{31}} \left\|\theta - \overline{\theta}^Z\right\|^2_{L^2(0,T;L^2(\Omega))} \\ + \left\|\chi - \widehat{\chi}^Z\right\|^2_{L^2(0,T;H^1(\Omega))\cap C([0,T];L^2(\Omega))} \le C_{10} A^Z + C_{32} |Z|^2. \end{aligned} \tag{6.40}$$

Therefore, by comparing the terms in (6.19), and using (6.10) and (6.3), we get

$$\left\| c_0 \left(\theta - \widehat{\theta}^Z\right) + \lambda(\chi) - \widehat{\lambda}^Z \right\|^2_{L^\infty(0,T;H^1(\Omega)^*)} \le C_{33} \left(A_{13} + |Z|^2\right). \tag{6.41}$$

Now, **(A2)**, (6.18), (6.4), (6.6), (6.7), $L^{\frac{5}{3}}(\Omega) \subset H^1(\Omega)^*$, (6.2a), and (2.1a) yield that

$$\left\| \theta - \widehat{\theta}^Z \right\|^2_{C([0,T];H^1(\Omega)^*)} \le C_{34} \left(A_{13} + |Z|^2\right). \tag{6.42}$$

Thanks to (6.17) and Young's inequality, we deduce

$$C_{10} A^Z \le \frac{1}{4C_{31}} \left\| u - \overline{u}^Z \right\|^2_{L^2(0,T;L^2(\Omega))} + C_{35} |Z|^{\frac{40}{23}}.$$

Hence, (6.40), (6.42), and (6.5) yield that (2.21) holds with $|Z|$ replaced by $|Z|^{\frac{20}{23}}$.
If we assume that at least one of the assumptions $\Omega \subset \mathbb{R}^2$ or $\lambda'_d = \lambda'_*$ is satisfied, applying (6.16), Schwarz's inequality, (4.48), (6.7), (6.6), and Young's inequality leads to

$$C_{10} A^Z \le C_{36} |Z| \left\| u - \overline{u}^Z \right\|_{L^2(0,T;L^2(\Omega))} \le C_{37} |Z|^2 + \frac{1}{4C_{31}} \left\| u - \overline{u}^Z \right\|_{L^2(0,T;L^2(\Omega))}.$$

Combining this with (6.40), (6.42), and (6.5), we see that (2.21) is satisfied. $\square$

A Appendix

For convenience, we list some inequalities and equalities used throughout this paper.

Lemma AP.1 (Young's inequality). *For $a, b \in \mathbb{R}$, $\sigma > 0$, $p > 1$, $q := \frac{p}{p-1}$, it holds*

$$ab \le \frac{1}{p} |a|^p + \frac{1}{q} |b|^q, \quad ab \le \frac{1}{p} \sigma^{-(p-1)} |a|^p + \frac{1}{q} \sigma |b|^q,$$
$$|a|^{ps} |b|^{p(1-s)} \le s \left(\frac{\sigma}{1-s}\right)^{\frac{s-1}{s}} |a|^p + \sigma |b|^p, \quad \forall\, 0 < s < 1.$$

Lemma AP.2 (Generalized Hölder's inequality). *For a bounded, open domain $\Omega \subset \mathbb{R}^N$ with $N \in \mathbb{N}$, $p, p_1, p_2, p_3 \in [1, \infty]$, $f_1 \in L^{p_1}(\Omega)$, $f_2 \in L^{p_2}(\Omega)$, and $f_3 \in L^{p_3}(\Omega)$ such that*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{p},$$

we have $f_1 \cdot f_2 \cdot f_3 \in L^p(\Omega)$ and

$$\|f_1 \cdot f_2 \cdot f_3\|_{L^p(\Omega)} \le \|f_1\|_{L^{p_1}(\Omega)} \|f_2\|_{L^{p_2}(\Omega)} \|f_3\|_{L^{p_3}(\Omega)}.$$

Thanks to Sobolev's embedding Theorem, we have

Lemma AP.3. *For a bounded, open domain $\Omega \subset \mathbb{R}^N$ with $N \in \{2, 3\}$ and Lipschitz boundary, there is a positive constant C such that*

$$\|v^p\|_{L^{\frac{6}{p}}(\Omega)} = \|v\|^p_{L^6(\Omega)} \le C^p \|v\|^p_{H^1(\Omega)}, \quad \forall\, v \in H^1(\Omega),\ p \in (0, 6]. \tag{AP.1}$$

The following classical elliptic estimate can be found in [Ama93, Remark 9.3 d].

Lemma AP.4. *For a bounded, open domain Ω with $\partial\Omega$ smooth there is a positive constant C such that*

$$\|v\|_{H^2(\Omega)}^2 \le C\left(\|\Delta v\|_{L^2(\Omega)}^2 + \left\|\frac{\partial v}{\partial n}\right\|_{H^{\frac{1}{2}}(\Gamma)}^2 + \|v\|_{L^2(\Omega)}^2\right), \quad \forall\, v \in H^2(\Omega).$$

In particular, for all $v \in H^2(\Omega)$ with $\frac{\partial v}{\partial n} = 0$ a.e. on Γ,

$$\|v\|_{H^2(\Omega)}^2 \le C\left(\|\Delta v\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2\right).$$

The following version of the Gagliardo–Nirenberg inequality is a special case of those considered in [Zhe95, Th. 1.1.4ii]

Lemma AP.5. *Let $\Omega \subset \mathbb{R}^N$ with $N \in \{2,3\}$ be a bounded domain with $\partial\Omega$ smooth. Let $2 < p < 6$ be given and $a := \frac{3}{2} - \frac{3}{p}$, Then there is a positive constant C such that*

$$\|u\|_{L^p(\Omega)} \le C\,\|u\|_{H^1(\Omega)}^a\,\|u\|_{L^2(\Omega)}^{1-a}, \quad \|u\|_{L^p(\Omega)} \le C\,\|u\|_{H^2(\Omega)}^{\frac{a}{2}}\,\|u\|_{L^2(\Omega)}^{1-\frac{a}{2}}.$$

If $\Omega \subset \mathbb{R}^2$, then the first estimate is also satisfied for $a = 1 - \frac{2}{p}$.

Elementary calculations lead to

Lemma AP.6. *For $n \in \mathbb{N}$, $a_0, a_1, \dots, a_n$, $b_0, b_1, \dots, b_n \in \mathbb{R}$, we have*

$$\sum_{i=1}^n a_i \sum_{j=1}^i b_j = \left(\sum_{i=1}^n a_i\right)\left(\sum_{i=1}^n b_i\right) - \sum_{j=1}^{n-1} b_{j+1} \sum_{i=1}^j a_i, \tag{AP.2}$$

$$\sum_{i=1}^n a_i \sum_{j=1}^i a_j = \frac{1}{2}\left(\sum_{i=1}^n a_i\right)^2 + \frac{1}{2}\sum_{i=1}^n a_i^2, \tag{AP.3}$$

$$\sum_{i=1}^n a_i(b_i - b_{i-1}) = a_n b_n - a_1 b_0 - \sum_{i=1}^{n-1}(a_{i+1} - a_i)\, b_i. \tag{AP.4}$$

Lemma AP.7. *Let H be a Hilbert space with scalar-product $\langle\cdot,\cdot\rangle_H$ and norm $\|\cdot\|_H$. Then we have*

$$\langle a, a-b\rangle_H = \frac{1}{2}\|a\|_H^2 - \frac{1}{2}\|b\|_H^2 + \frac{1}{2}\|a-b\|_H^2, \quad \forall\, a, b \in H. \tag{AP.5}$$

The next lemma follows form elementary analysis.

Lemma AP.8. *Let $a, b > 0$ be given. Then there exists a constant $C > 0$, such that*

$$\frac{a}{2}s + b\,|\ln s| \le as - b\ln s + C, \quad \forall\, s > 0.$$

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